ASEN 5327 Computational Fluid Dynamics
Spring 2009

Characteristic Boundary Conditions for the Quasi-1D System

The quasi-one-dimensional Euler equations can be written as

$$\frac{\partial}{\partial t}(UA) + \frac{\partial}{\partial x}(FA) = Q \tag{1}$$

where

$$U = \left\{ \begin{array}{c} \rho \\ \rho u \\ E_t \end{array} \right\} \qquad F = \left\{ \begin{array}{c} \rho u \\ \rho u^2 + p \\ (E_t + p)u \end{array} \right\} \qquad Q = \left\{ \begin{array}{c} 0 \\ p\frac{dA}{dx} \\ 0 \end{array} \right\} \tag{2}$$

and where $A(x)$ is the cross-sectional area. In deriving the characteristic form, it is useful to define the "primitive variable" solution vector as

$$V = \left\{ \begin{array}{c} \rho \\ u \\ p \end{array} \right\} \tag{3}$$

Starting from Eq. (1) we manipulate as follows

$$\begin{aligned}
\frac{\partial}{\partial t}(UA) + \frac{\partial}{\partial x}(FA) &= Q \\
A\frac{\partial U}{\partial t} + A\frac{\partial F}{\partial x} &= Q - F\frac{dA}{dx} \\
\left[\frac{\partial U}{\partial V}\right]\frac{\partial V}{\partial t} + \left[\frac{\partial F}{\partial V}\right]\frac{\partial V}{\partial x} &= \frac{1}{A}\left(Q - F\frac{dA}{dx}\right) \\
\frac{\partial V}{\partial t} + \underbrace{\left[\frac{\partial U}{\partial V}\right]^{-1}\left[\frac{\partial F}{\partial V}\right]}_{[B]}\frac{\partial V}{\partial x} &= \underbrace{\left[\frac{\partial U}{\partial V}\right]^{-1}\frac{1}{A}\left(Q - F\frac{dA}{dx}\right)}_{\tilde{Q}} \\
\frac{\partial V}{\partial t} + [B]\frac{\partial V}{\partial x} &= \tilde{Q}
\end{aligned} \tag{4}$$

Let the eigenvectors of $[B]$ be denoted as $[T]$. It is then possible to write

$$[T]^{-1}[B][T] = [\Lambda] \tag{5}$$

where $[\Lambda]$ is the diagonal matrix of eigenvalues.

We can use the above result to diagonalize Eq. (4) as follows

$$
\begin{aligned}
\frac{\partial V}{\partial t} + [B]\frac{\partial V}{\partial x} &= \tilde{Q} \\
[T]^{-1}\frac{\partial V}{\partial t} + \underbrace{[T]^{-1}[B][T]}_{[\Lambda]}[T]^{-1}\frac{\partial V}{\partial x} &= \underbrace{[T]^{-1}\tilde{Q}}_{\hat{Q}} \qquad (6) \\
[T]^{-1}\frac{\partial V}{\partial t} + [\Lambda][T]^{-1}\frac{\partial V}{\partial x} &= \hat{Q} \qquad (7)
\end{aligned}
$$

After completing all of the matrix operations indicated above, the characteristic equations in component form are

$$
\begin{aligned}
\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) - \frac{1}{c^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) &= 0 \qquad (8) \\
\left(\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x}\right) + \frac{1}{\rho c}\left(\frac{\partial p}{\partial t} + (u+c)\frac{\partial p}{\partial x}\right) &= -\left(\frac{uc}{A}\right)\frac{dA}{dx} \qquad (9) \\
\left(\frac{\partial u}{\partial t} + (u-c)\frac{\partial u}{\partial x}\right) - \frac{1}{\rho c}\left(\frac{\partial p}{\partial t} + (u-c)\frac{\partial p}{\partial x}\right) &= \left(\frac{uc}{A}\right)\frac{dA}{dx} \qquad (10)
\end{aligned}
$$

Consider a subsonic exit boundary. The eigenvalues u and $u+c$ are positive, whereas the eigenvalue $u-c$ is negative. Thus two pieces of information propagate downstream, from within the domain to the exit, and one piece of information propagates upstream from outside the domain. We must solve Eqs. (8) and (9) at the exit in order to capture the downstream propagation. The upstream propagation can be accomplished by replacing Eq. (10) with the condition $(\partial p/\partial t)_{exit} = 0$.

We can either time advance Eqs. (8) and (9) at the boundary as they appear (in primitive variable form), or create an effective right hand side for the conservative variables and use this to update the exit boundary at the same time as the interior is updated. This latter approach may be preferable since this automatically time advances the exit plane with the same scheme used for the interior. The obvious connection between the rates of change of the conservative and primitive variables is

$$
\frac{\partial U}{\partial t} = \left[\frac{\partial U}{\partial V}\right]\frac{\partial V}{\partial t} \qquad (11)
$$

1 Appendix A: Required Matrices

It is a straightforward matter to derive the following matrices:

$$\left[\frac{\partial U}{\partial V}\right] = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{1}{2}u^2 & \rho u & \frac{1}{\gamma-1} \end{bmatrix} \qquad \left[\frac{\partial U}{\partial V}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 \\ \left(\frac{\gamma-1}{2}\right)u^2 & -(\gamma-1)u & \gamma-1 \end{bmatrix}$$

$$\left[\frac{\partial F}{\partial V}\right] = \begin{bmatrix} u & \rho & 0 \\ u^2 & 2\rho u & 1 \\ \frac{1}{2}u^3 & \left(\frac{\gamma}{\gamma-1}\right)p + \frac{3}{2}\rho u^2 & \left(\frac{\gamma}{\gamma-1}\right)u \end{bmatrix} \qquad [B] \equiv \left[\frac{\partial U}{\partial V}\right]^{-1}\left[\frac{\partial F}{\partial V}\right] = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{bmatrix}$$

$$[\Lambda] = \begin{bmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{bmatrix} \qquad [T] = \begin{bmatrix} 1 & \rho & \rho \\ 0 & c & -c \\ 0 & \rho c^2 & \rho c^2 \end{bmatrix} \qquad [T]^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{c^2} \\ 0 & \frac{1}{2c} & \frac{1}{2\rho c^2} \\ 0 & -\frac{1}{2c} & \frac{1}{2\rho c^2} \end{bmatrix}$$