ASEN 5327 Computational Fluid Dynamics
Spring 2009

Homework 1, due Thursday, January 29

Problems from Tannehill, Anderson, and Pletcher
2.10
In problem the second equation should read

$$\frac{\partial v}{\partial t} + 2\frac{\partial u}{\partial x} = 0$$

2.13
2.14
2.15
2.16
2.20
For a more interesting solution to this problem, consider the following initial conditions

$$u(x, y = 0) = \sin^2(x)$$
$$u_y(x, y = 0) = \sin(x)$$

1. The energy equation in fluid mechanics can be written in various forms. In terms of the temperature, the equation is

$$\rho c_v \left[\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right] + p\frac{\partial u_j}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j} \right) + \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

where c_v is the constant volume specific heat (a constant), k is the thermal conductivity (a function of temperature) and t_{ij} is the viscous stress, defined as

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

Here μ is the viscosity coefficient (a function of temperature) and $\delta_{ij} =$ is the Kronecker delta (equal 1 for $i = j$, 0 otherwise).

(a) Attempt to write the temperature equation in strong conservation law form. You may use any other valid conservation law in performing this derivation.

(b) Identify any spatial terms that can not be put in divergence form.

(c) Form a "kinetic energy transport equation" by dotting the momentum equation with the velocity vector. Start with the momentum equation in the following form

$$\rho\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]+\frac{\partial p}{\partial x_i}=\frac{\partial \tau_{ij}}{\partial x_j}$$

(d) Can the kinetic energy transport equation be put in strong conservation law form? If not, identify the terms which prohibit this.

(e) By adding the temperature and kinetic energy equations, show that the resulting equation for the total energy, $E_t=\rho(c_vT+1/2u_iu_i)$ can be put in strong conservation law form.

(f) Which form of the energy equation should be used in numerical simulations? Why is this the case?

2. Burgers equation is a commonly-used model equation for non-linear advection. It reads

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$$

Consider a problem on a domain $0\le x\le L$.

(a) Show that the average value of u satisfies the following global conservation constraint

$$L\frac{d\bar{u}}{dt}=\frac{1}{2}u(x=0,t)^2-\frac{1}{2}u(x=L,t)^2$$

where

$$\bar{u}(t)=\frac{1}{L}\int_0^L u\;dx$$

(b) Consider the following discretization of Burgers equation

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+u_i^n\left(\frac{u_i^n-u_{i-1}^n}{\Delta x}\right)=0$$

on a grid defined by $x_i=i\Delta x$, $i=0,1,2,...N$, $\Delta x=L/N$, where x_0 and x_N are known boundary values. By making use of the following discrete integration rule,

$$\int_0^L u(\xi,t)d\xi\simeq\sum_{i=1}^{N}u_i^n\Delta x$$

show that the above discretization rule does not satisfy the global conservation constraint.

(c) Show that the Burgers equation can also be written in the following "strong conservation law" form

$$\frac{\partial u}{\partial t}+\frac{\partial(u^2/2)}{\partial x}=0$$

(d) Show that the following discretization

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{2}\left(\frac{(u_i^n)^2 - (u_{i-1}^n)^2}{\Delta x}\right) = 0$$

of the strong conservation law form does satisfy the global conservation constraint.

3. Write a computer program to solve the 1-D wave equation as described on the "animations/wave equation example" link on the course website. Use the generalized approach to the spatial derivative as shown on the website, using the parameters α, β, and γ. Run three cases, one for a forward difference, one for a central difference, and one for a backward difference. In each case, use 100 points in x and a CFL number of 0.1. Plot the computed solution against the exact solution at a time of $t = 0.5L/c$ for cases where the solution does not blow up. For cases where the solution blows up, plot the average and root mean square of the solution as a function of time. Use a log scale as appopriate if the numbers become extremely large.