ASEN 5327 Computational Fluid Dynamics
Spring 2009

Homework 2, due Thursday, February 19

Problems from Tannehill, Anderson, and Pletcher
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1. In trying to improve the accuracy of a numerical solution, one can increase the order of accuracy of the finite-difference operators themselves, or one can simply refine the mesh. The effective wavenumber analysis can offer valuable insight into the tradeoffs of these two approaches.

(a) Starting with the finite-difference expression for a 4th order central difference, derive the following result for the corresponding effective wavenumber

$$k_{eff_4} = \frac{8\sin(k\Delta x) - \sin(2k\Delta x)}{6\Delta x}$$

As derived in lecture and in the textbook, the analogous result for a 2nd order central difference is

$$k_{eff_2} = \frac{\sin(k\Delta x)}{\Delta x}$$

(b) Consider a situation where 2nd order central differences are used on a mesh containing N points. In one approach to improve accuracy, the 2nd order differences are replaced with 4th order differences while the number of mesh points is held fixed at N. In a second approach, the 2nd order differences are retained, but the number of mesh points is increased to γN ($\gamma \geq 1$). Define the non-dimensional finite-difference error as $E\Delta x_0/\pi = (k - k_{eff})\Delta x_0/\pi$. Plot this error for the 4th order scheme over the non-dimensional wavenumber range $0 \leq k\Delta x_0/\pi \leq 1$ (since $0 \leq k \leq N/2$ and $\Delta x_0 = 2\pi/N$). On the same plot, display the second order error applied to a mesh with $2N$ points. How do the two errors compare at the low and high wavenumber ends of the spectrum? Which approach appears to be better? Is this conclusion problem-dependent?

2. The 1-D advection-diffusion equation is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)$$

Where c and ν are constants.

(a) Consider a problem on a domain $0 \le x \le L$ with Dirchlet boundary conditions $u(x=0,t) = 0, \quad u(x=L,t) = u_1$. Show that the advection-diffusion equation and boundary conditions can be put into the following non-dimensional form

$$\frac{\partial u^*}{\partial t^*} + \frac{\partial u^*}{\partial x^*} = \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}}$$

$$u^*(x^*=0,t^*) = 0, \qquad u^*(x^*=1,t^*) = u_1^*$$

Where $x^* = x/L, \quad u^* = u/c, \quad t^* = ct/L, \quad$ and $Re = cL/\nu$.

(b) Notice that the advection-diffusion equation can admit steady state solutions. Show that a steady state solution to the non-dimensional problem is

$$u_\infty^*(x^*) \equiv u^*(x^*, t^* \to \infty) = \left(\frac{\exp(x^* Re) - 1.0}{\exp(\quad Re) - 1.0}\right) u_1^*$$

(c) Attempt to determine a general solution to the time-dependent problem started from the initial condition $u^*(x^*, t^*=0) = u_0^*(x^*)$.

(d) Modify your wave equation program to solve the advection-diffusion equation as stated above in non-dimensional form. Generalize the program so that the mesh can be compressed near the $x^* = 1$ boundary. Do this by making use of the following geometric stretching law

$$\Delta x_{i+1} = (1+\sigma)\Delta x_i$$

where σ is the constant mesh expansion rate. Positive values of σ produce meshes that expand in the direction of the index i, negative values result in compression. As an example, a value of $\sigma = 0.1$ results in a mesh where each successive cell is 10% larger than its neighbor to the left. Notice that one must specify either the expansion rate or the spacing at one end of the distribution if a set number of points is to cover a domain of fixed length.

As we will derive later in the course, the maximum allowable time step for our chosen discretization of the advection-diffusion equation is predicted to be

$$\Delta t^* = \min(\ 2/Re,\ 0.5Re(\Delta x^*_{min})^2\ )$$

for a central difference of the advective term and

$$\Delta t^* = \frac{Re(\Delta x^*_{min})^2}{2 + Re\Delta x^*_{min}}$$

for a first order upwind difference. In these equations Δx^*_{min} is the minimum mesh spacing within the domain. Note that these estimates are only apporximate and thus it may be necessary to lower the time step somewhat in order to realize stable solutions.

The steady-state result can be achieved by simply time-advancing the solution until it ceases to change. This should happen by $t^* = 5$.

Use your program to determine a steady-state solution for $Re = 40$, subject to the following initial and boundary conditions

$$u_0^*(x^*) = 2x^*, \qquad u_1^* = 2$$

Do this first with a uniform mesh and use both central and backward differences for the advective term. Use a central difference for the diffusive term. Consider meshes with 10, 20, 40, 80, and 160 points. Compute the rms error between the computed and exact solutions at a time of $t^* = 5$ and use these data to make a convergence plot. Confirm the orders of accuracy of the central and backward difference schemes. Also plot the solutions at $t^* = 5$ along with the exact steady-state solution for the cases with 10, 40, and 160 grid points.

(e) Investigate the benefit of mesh compression by running cases with 20 points and compression rates of σ = 0.0, -0.05, -0.10, -0.15, and -0.20 for both backward and central differences applied to the advective term. Make a plot where the rms error (measured at $t^* = 5$) is plotted as a function of the mesh stretching factor ($|\sigma|$). Also plot the solutions at $t^* = 5$ along with the exact steady-state solution for the cases with σ = 0.0 and -0.15.