ASEN 5327 Computational Fluid Dynamics
Spring 2009

Homework 3, due Tuesday, March 10

Problems from Tannehill, Anderson, and Pletcher
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1. As demonstrated in your prior homework assignment, the advection-diffusion equation has a stability bound that is dependent on Reynolds number as well as the details of the time-advancement scheme. In this exercise and the following two, we will use both analysis and simulation to determine and verify the stability characteristics of numerical approximations to this equation.

(a) Assuming a uniform mesh with periodic boundary conditions, determine the eigenvalue spectrum when both the advective and diffusive terms are approximated with 3-point central differences. Plot the eigenvalue spectrum (multiplied by Δt) in the complex plane for R_Δ=1.0, 2.0, 3.0, where $R_\Delta = c\Delta x/\nu$ is the mesh Reynolds number. Choose an appropriate value for the CFL number (CFL=$c\Delta t/\Delta x$) for each value of R_Δ so that the minimum real value of $\lambda\Delta t$ is -2.0 in each case. Superimpose the stability boundary for the Explict Euler scheme on this plot.

(b) Based on the plot made above, deduce that the simple scaling employed is sufficient to determine the stability bound when $R_\Delta \leq 2.0$. Write down the corresponding relationship in the form $CFL_{max} = f(R_\Delta)$. Also show that further analysis must be performed in order to determine the stability bound for $R_\Delta > 2.0$.

(c) From the plot made above, we see that the eigenvalues break through the stability boundary near the origin when $R_\Delta > 2.0$. We can therefore analyze this problem by developing the expression for the amplification factor in a series expansion taken about $k\Delta x = 0$. Undertake such an analysis where the stability requirement for the Explicit Euler method, $|e^{\lambda\Delta t}| = |1 + \lambda\Delta t| \leq 1$, is expanded for small $k\Delta x = 0$. Show that this analysis leads to a second constraint of the form $CFL_{max} = f(R_\Delta)$, which must be respected when $R_\Delta > 2.0$.

(d) Show that the results from parts (b) and (c) can be combined using a min function in order to arrive at a unified expression for the maximum allowable CFL number as a function of the mesh Reynolds number (R_Δ). What is the asymptotic value of CFL_{max} in the inviscid limit (i.e. as $R_\Delta \to \infty$). Argue convincingly that this is the expected result.

2. Repeat the above analysis for a first-order upwind approximation of the advective term (while retaining the second order centered difference for the diffusion term).

1. Plot the eigenvalue spectra for $R_\Delta = 1.0, 2.0, 10.0$.
2. Determine the time step restriction in the form $CFL_{max} = f(R_\Delta)$.
3. What is the asymptotic value of CFL_{max} in the inviscid limit for this scheme. Argue convincingly that this is the expected result.

3. Check the results of problems 1 and 2 by performing numerical simulations with your advection-diffusion solver. Use 40 equi-spaced points in x and adjust the Reynolds number so that you achieve cases with $R_\Delta = 0.5, 2.0, 8.0$. For each case, adjust the time step until the solution is diverging when taken out to $ct/L = 5.0$. A good way to assess this is to compute the rms difference between the computed and steady-state solutions at each time step. The rms difference should settle to a constant value for stable cases. Compare the observed stability bound with the prediction. Comment on any differences.