ASEN 5327 Computational Fluid Dynamics

Answers to Practice Midterm Exam

1. The character of the following equation depends on the numerical value of the parameter c

$$u_{xx} + 2u_{xy} + cu_{yy} - 3u_x - 4u_y - u = 0$$

(a) (10 pts.) For each equation type (hyperbolic, parabolic, elliptic), determine the corresponding range of values for c, as well as the equations for the characteristic directions (i.e. $\xi = \xi(x, y)$, $\eta = \eta(x, y)$).

ans:

$$\begin{array}{lll} \text{hyperbolic}: & c < 1 & \xi = y - (1 + \sqrt{1-c})x, \quad \eta = y - (1 - \sqrt{1-c})x \\ \text{parabolic}: & c = 1 & \xi = y - x \\ \text{elliptic}: & c > 1 & \text{no characteristic directions} \end{array}$$

(b) (5 pts.) Rewrite the equation as a coupled system of first order equations. Your answer should be in the following form

$$[A]\frac{\partial \vec{U}}{\partial x} + [B]\frac{\partial \vec{U}}{\partial y} = \vec{S}$$

where $[A]$ and $[B]$ are the coupling matrices, $\vec{U}$ is the solution vector, and $\vec{S}$ are the source terms.

ans:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{d}{dx}\left\{ \begin{array}{c} v \\ w \end{array} \right\} + \begin{bmatrix} 2 & c \\ -1 & 0 \end{bmatrix} \frac{d}{dy}\left\{ \begin{array}{c} v \\ w \end{array} \right\} = \left\{ \begin{array}{c} 3v + 4w + u \\ 0 \end{array} \right\}$$

(c) (10 pts.) Multiply the solution form found in part (b) by $[A]^{-1}$ in order to put it in the following form

$$\frac{\partial \vec{U}}{\partial x} + [D]\frac{\partial \vec{U}}{\partial y} = \vec{R}$$

where $[D] = [A]^{-1}[B]$, $\vec{R} = [A]^{-1}\vec{s}$. Determine the character of this system as a function of the value of c and show that the results are identical to those found in part (a).

ans:

$$\frac{d}{dx}\left\{ \begin{array}{c} v \\ w \end{array} \right\} + \begin{bmatrix} 2 & c \\ -1 & 0 \end{bmatrix} \frac{d}{dy}\left\{ \begin{array}{c} v \\ w \end{array} \right\} = \left\{ \begin{array}{c} 3v + 4w + u \\ 0 \end{array} \right\}$$

The eigenvalues of the coupling matrix are

$$\lambda = 1 \pm \sqrt{1-c}$$

2. Consider the following two alternative difference approximations to the first derivative

$$\frac{\partial u}{\partial x} \simeq \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x} \tag{1}$$

$$\frac{\partial u}{\partial x} \simeq \frac{-u_{i+2} + 6u_{i+1} - 3u_i - 2u_{i-1}}{6\Delta x} \tag{2}$$

(a) (12 pts.) Determine the order of accuracy for each of these difference approximations and write out the leading leading term of the truncation error in each case.

ans:

$$\frac{\partial u}{\partial x} = \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x} + \frac{\Delta x^4}{30}\frac{\partial^5 u}{\partial x^5} + \ldots \tag{3}$$

$$\frac{\partial u}{\partial x} = \frac{-u_{i+2} + 6u_{i+1} - 3u_i - 2u_{i-1}}{6\Delta x} + \frac{\Delta x^3}{12}\frac{\partial^4 u}{\partial x^4} + \ldots \tag{4}$$

(b) (8 pts.) Consider use of the above difference approximation for the solution of the wave equation

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0; \qquad c > 0$$

Using a explicit Euler time advancement scheme, write down the modified equations that result from each of the two difference approximations. Comment on how the modified terms will alter the physics in each case.

ans:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - c\left(\frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x}\right) = \underbrace{\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}}_{\text{diffusion}} + \underbrace{\frac{c\Delta x^4}{30}\frac{\partial^5 u}{\partial x^5}}_{\text{dispersion}} + \ldots$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - c\left(\frac{-u_{i+2} + 6u_{i+1} - 3u_i - 2u_{i-1}}{6\Delta x}\right) = \underbrace{\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}}_{\text{diffusion}} + \underbrace{\frac{c\Delta x^3}{12}\frac{\partial^4 u}{\partial x^4}}_{\text{diffusion}} + \ldots$$

(c) (5 pts.) Write down the two time advancement schemes corresponding to each of the difference approximation applied to the wave equation as discussed above. Present the results in the form

$$u_i^{n+1} = {u_i}^n + \ldots$$

ans:

$$u_i^{n+1} = {u_i}^n + \left(\frac{c\Delta t}{\Delta x}\right)\left(\frac{-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n}{12}\right)$$

$$u_i^{n+1} = {u_i}^n + \left(\frac{c\Delta t}{\Delta x}\right)\left(\frac{-u_{i+2}^n + 6u_{i+1}^n - 3u_i^n - 2u_{i-1}}{6}\right)$$

3. Now Consider the stability properties of the schemes found in part (c) of the previous problem.

(a) (10 pts.) Determine the stability bound, if any, on the first scheme constructed in part (c) of the previous problem.

ans: Unconditionally unstable.

(b) (15 pts.) Determine the stability bound, if any, on the second scheme constructed in part (c) of the previous problem.

ans: Unconditionally unstable.