2.10 −7

$$u_t + 8v_x = 0$$

$$v_t + 2u_x = 0$$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} = 0$$

eigenvalues of the coupling matrix are

$$\begin{vmatrix} -\lambda & 8 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 16 = 0$$

$$\lambda = \pm 4$$

Thus the system is hyperbolic

2.13 $u_{tt} + u_{xx} + u_x = -e^{-kt}$ 

$a = 1 \quad b = 0 \quad c = 1$

$b^2 - 4ac = 0 - 4(1)(1) = -4$

thus the equation is elliptic.

$u_{xx} - u_{xy} + u_y = 4$

$a = 1 \quad b = -1 \quad c = 0$

$b^2 - 4ac = (-1)^2 - 4(1)(0) = 1$

thus the equation is hyperbolic.

2.14 $$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{[A]} \frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{[B]} \frac{d}{dy}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

eigenvalues of $[A]$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

elliptic in $(x,t)$

eigenvalues of $[B]$

$$\begin{vmatrix} (-1-\lambda) & 0 \\ 0 & (1-\lambda) \end{vmatrix} = 0$$

$$-(\lambda+1)(\lambda-1) - 0 = 0$$

$$-(\lambda^2 - 1) = 0$$

$$\lambda = \pm 1$$

hyperbolic in $(y,t)$

2.15 $f(x) = \sin(x) \quad 0 < x < \pi$

a) $f(x) = \sum_{n=0}^{\infty} c_n \cos(nx)$

where $c_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$

$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$

$$c_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)dx = \frac{-1}{\pi}\cos(x)\Big|_0^{\pi} = \frac{-1}{\pi}\left(-1-1\right) = \frac{2}{\pi}$$

$$c_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)dx$$

$$= -\frac{2}{\pi}\left[\frac{\cos(1-n)x}{2(1-n)} + \frac{\cos(1+n)x}{2(1+n)}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{(n+1)\cos(n-1)x - (n-1)\cos(n+1)x}{2(1-n^2)}\right]_0^{\pi}$$

$$= \frac{1}{\pi(n^2-1)}\Big[n\left(\cos(n-1)x - \cos(n+1)x\right) + \left(\cos(n-1)x - \cos(n+1)x\right)$$

$$= \frac{1}{\pi(n^2-1)}\Big[n\left(\cos nx\cos x + \sin nx\sin x - \cos nx\cos x + \sin nx\sin x\right) + \left(\cos nx\cos x + \sin nx\sin x + \cos nx\cos x - \sin nx\sin x\right)\Big]_0^{\pi}$$

$$= \frac{2}{\pi(n^2-1)}\Big[n\sin nx\sin x + \cos nx\cos x\Big]_0^{\pi}$$

$$= \frac{2}{\pi(n^2-1)}\left[\left(\pi \cancel{\sin n\pi}^{\,0} \sin\pi - 0\right) + \cos n\pi \cos\pi - \cos(0)\cos(0)\right]$$

$$= \frac{2}{\pi(n^2-1)}\left[(-1)^n(-1) - 1\right]$$

$$= \frac{2}{\pi(n^2-1)}\left[(-1)^{n+1} - 1\right]$$

$$\boxed{C_n = \begin{cases} \dfrac{-4}{\pi(n^2-1)} & n \text{ even} \\ 0 & n \quad \text{odd} \end{cases}}$$

b)

$$\left.\begin{array}{l} C_0 = 0 \\ C_1 = 1 \\ C_n = 0 \quad n > 1 \end{array}\right\} \text{by inspection}$$

2.16 

a) $U_{xx} + 3U_{xy} + 2U_{yy} = 0$

$a = 1 \quad b = 3 \quad c = 2$

$b^2 - 4ac = 9 - 4(1)(2) = 1$

hyperbolic equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm 1}{2} = 2, 1$$

$$\boxed{\xi = y - 2x \qquad \eta = y - x}$$

b) $U_{xx} - 2U_{xy} + U_{yy} = 0$

$a = 1 \quad b = -2 \quad c = 1$

$b^2 - 4ac = 4 - 4(1)(1) = 0$

Parabolic

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2}{2(1)} = -1$$

$$\boxed{\xi = y + x}$$

Choose $\eta$ to be orthogonal to $\xi$

$$\boxed{\eta = y - x}$$

2.20

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad y \geq 0$$

$$u(x,0) = 1$$
$$u_y(x,0) = 0$$

Following Example 2.5 on P. 26, the solution is

$$u(x,y) = \frac{f(x+y) + f(x-y)}{2} + \frac{1}{2}\int_{x-y}^{x+y} g(\eta)\,d\eta$$

where

$$f(x) = u(x,0) = 1$$
$$g(x) = u_y(x,0) = 0$$

Using the given initial conditions

$$u(x,y) = \frac{1+1}{2} + \frac{1}{2}\int_{x-y}^{x+y} 0\,d\eta$$

$$\boxed{u(x,y) = 1}$$

1. Prior to working out the solution to this problem, it is useful to establish the connection between the material derivative and a divergence form. Let $\alpha$ be a generic fluid property. Its material derivative can be manipulated as follows

$$\rho \frac{D\alpha}{Dt} = \rho\left(\frac{\partial \alpha}{\partial t} + u_j \frac{\partial \alpha}{\partial x_j}\right)$$

$$= \frac{\partial}{\partial t}\rho\alpha - \alpha\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\rho\alpha u_j - \alpha\frac{\partial}{\partial x_j}\rho u_j$$

$$= \frac{\partial}{\partial t}\rho\alpha + \frac{\partial}{\partial x_j}\rho\alpha u_j - \alpha\underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\rho u_j\right)}_{\text{mass conservation} \to 0}$$

$$\rho \frac{D\alpha}{Dt} = \frac{\partial}{\partial t}\rho\alpha + \frac{\partial}{\partial x_j}\rho\alpha u_j \qquad (1)$$

Note that the right hand side is in divergence form whereas the left hand side is not.

(a) The temperature equation is

$$C_v \rho \frac{DT}{Dt} + P\frac{\partial u_j}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right) + \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

Making use of Eq. (1) and noting that $C_v = \text{const}$, we can rewrite the above equation as

$$\frac{\partial}{\partial t}\left(\rho C_v T\right) + \frac{\partial}{\partial x_j}\left(\rho C_v T u_j\right) + P\frac{\partial u_j}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right) + \tau_{ij}\frac{\partial u_i}{\partial x_j} \qquad (2)$$

It is not productive to operate on the pressure and viscous terms as these clearly can not be put in divergence form.

(b) The terms

$$P\frac{\partial u_j}{\partial x_j} \quad \text{and} \quad \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

Can not be put in divergence form.

(c) An equation for the Kinetic energy can be formed by dotting the momentum equation with the velocity

$$\rho\left( u_i \frac{\partial u_i}{\partial t} + u_i u_j \frac{\partial u_i}{\partial x_j} \right) + u_i \frac{\partial P}{\partial x_i} = u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\rho\left( \frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) + u_j \frac{\partial}{\partial x_j}\left(\frac{u_i u_i}{2}\right)\right) + \frac{\partial}{\partial x_j} P u_j - P\frac{\partial u_j}{\partial x_j} =$$

$$\frac{\partial}{\partial x_j}\left(\tau_{ij} u_i\right) - \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

Define $u^2 \equiv u_i u_i$ and make use of Eq. (1) to get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u^2 u_j\right) + \frac{\partial}{\partial x_j}(P u_j) - P\frac{\partial u_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\tau_{ij} u_i\right) - \tau_{ij}\frac{\partial u_i}{\partial x_j} \qquad (3)$$

(d) The terms

$$-P\frac{\partial u_j}{\partial x_j} \quad \text{and} \quad -\tau_{ij}\frac{\partial u_i}{\partial x_j}$$

Can not be put in divergence form.

(e) Equations (2) and (3) can be added to give

$$\frac{\partial}{\partial t}\left[\rho\left(c_v T + \frac{1}{2}u^2\right)\right] + \frac{\partial}{\partial x_j}\left[\rho\left(c_v T + \frac{1}{2}u^2 + \frac{p}{\rho}\right)u_j\right] + \cancel{p\frac{\partial u_j}{\partial x_j}} - \cancel{p\frac{\partial u_j}{\partial x_j}} =$$

$$-\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + \cancel{\tau_{ij}\frac{\partial u_i}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(\tau_{ij}u_j\right) - \cancel{\tau_{ij}\frac{\partial u_i}{\partial x_j}}$$

Define the total energy as $E_t = \rho\left(c_v T + \frac{1}{2}u^2\right)$ then the above equation can be written as

$$\frac{\partial}{\partial t}E_t + \frac{\partial}{\partial x_j}(E_t + p)u_j = -\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\tau_{ij}u_j\right) \qquad (4)$$

which is in divergence form

(f) We should use this latter form in numerical simulations since a consistent discrete approximation to it will ensure that energy is conserved, irrespective of numerical errors.

2. Burgers equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad (1)$$

(a) This equation can be integrated over a domain extending from $0 \le x \le L$ as follows

$$\int_0^L \frac{\partial u}{\partial t}dx + \int_0^L u\frac{\partial u}{\partial x} = 0 \qquad (2)$$

The first integral is manipulated as follows

$$\int_0^L \frac{\partial u}{\partial t}dx = \frac{d}{dt}\int_0^L u\,dx = L\frac{d}{dt}\underbrace{\left(\frac{1}{L}\int_0^L u\,dx\right)}_{\bar{u}} = L\frac{d\bar{u}}{dt} \qquad (3)$$

where $\bar{u}$ is the average of $u$ within the domain. The second term in Eq. (2) is simplified via integration by parts

$$\int_0^L u\frac{\partial u}{\partial x}dx = uu\Big|_0^L - \int_0^L u\frac{\partial u}{\partial x}dx$$

or

$$\int_0^L u\frac{\partial u}{\partial x} = \frac{1}{2}\left(u^2(L,t) - u^2(0,t)\right) \qquad (4)$$

Combining Eqs. (2), (3) and (4) we have

$$L\frac{d\bar{u}}{dt} = \frac{1}{2}u^2(0,t) - \frac{1}{2}u^2(L,t) \qquad (5)$$

(b) Consider the following discretization of Burgers equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n\left(\frac{u_i^n - u_{i-1}^n}{\Delta x_i}\right) = 0 \qquad (6)$$

Application of a discrete integration to this equation leads to

$$\frac{1}{\Delta t}\left[\sum_{i=1}^{N} u_i^{n+1}\Delta x_i - \sum_{i=1}^{N} u_i^n \Delta x_i\right] + \sum_{i=1}^{N} u_i^n\left(\frac{u_i^n - u_{i-1}^n}{\Delta x_i}\right)\Delta x_i = 0 \qquad (7)$$

Noting that the average $\bar{u}$ is defined as

$$\bar{u}^n = \frac{1}{L}\sum u_i^n \Delta x_i$$

The above equation becomes

$$L\left(\frac{\bar{u}^{n+1} - \bar{u}^n}{\Delta t}\right) = -\sum_{i=1}^{N} u_i^n\left(u_i^n - u_{i-1}^n\right)$$

The left hand side is clearly a discrete approximation to the time derivative in Eq. (5). The right hand side of the above equation expands to

$$\sum_{i=1}^{N} u_i^n\left(u_i^n - u_{i-1}^n\right) = \left(u_1^2 - u_1 u_0 + u_2^2 - u_2 u_1 + \dots + u_N^2 - u_N u_{N-1}\right)^n$$

which is not a discrete approximation to the right hand side of Eq. (5). Thus Eq. (6) is not a conservative discrete form.

(c) Making use of the fact that

$$u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)$$

Burgers equation can also be written as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0 \qquad (8)$$

A discrete version of this equation is

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{1}{2}\left[\frac{(U_i^n)^2 - (U_{i-1}^n)^2}{\Delta x_i}\right] = 0$$

Application of the discrete integration rule leads to

$$L\left(\frac{\overline{U}_i^{n+1} - \overline{U}_i^n}{\Delta t}\right) = -\frac{1}{2}\sum_{i=1}^{N}\left[(U_i^n)^2 - (U_{i-1}^n)^2\right] \quad (9)$$

The sums on the right hand side can be manipulated as follows

$$\sum_{i=1}^{N}(U_i^n)^2 = \sum_{i=1}^{N-1}(U_i^n)^2 + (U_N^n)^2$$

$$\sum_{i=1}^{N}(U_{i-1}^n)^2 = \sum_{i=0}^{N-1}(U_i^n)^2 = \sum_{i=1}^{N-1}(U_i^n)^2 + (U_0^n)^2$$

Substitution of these results into Eq. (9) yields

$$L\left(\frac{\overline{U}_i^{n+1} - \overline{U}_i^n}{\Delta t}\right) = \frac{1}{2}(U_0^n)^2 - \frac{1}{2}(U_N^n)^2$$

which is a valid discrete analog of Eq. (5).

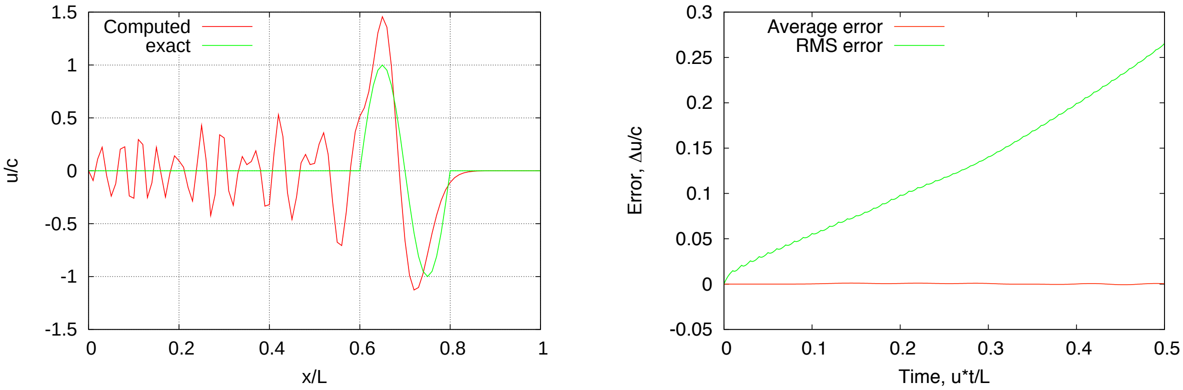


Figure 1: Second order central difference in space, explicit Euler in time. Nx=100, CFL=0.1.

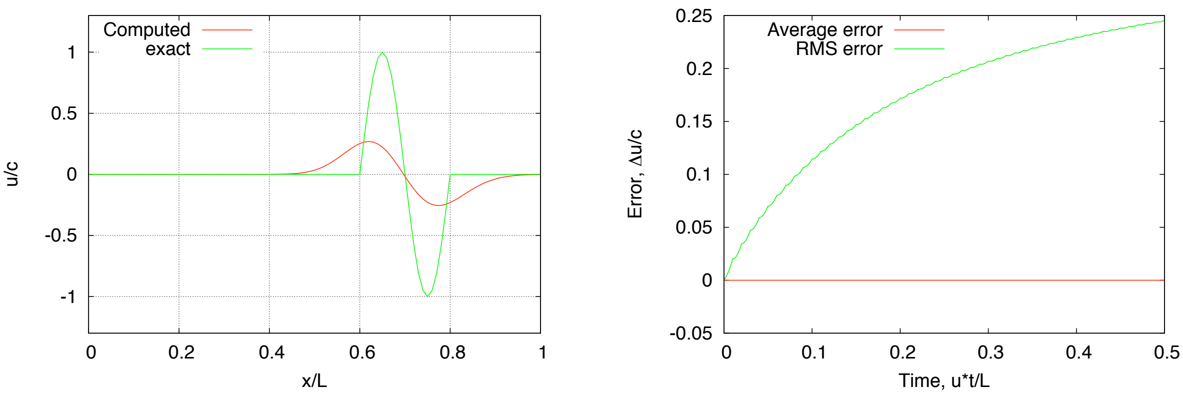


Figure 2: First order upwind in space, explicit Euler in time. Nx=100, CFL=0.1.

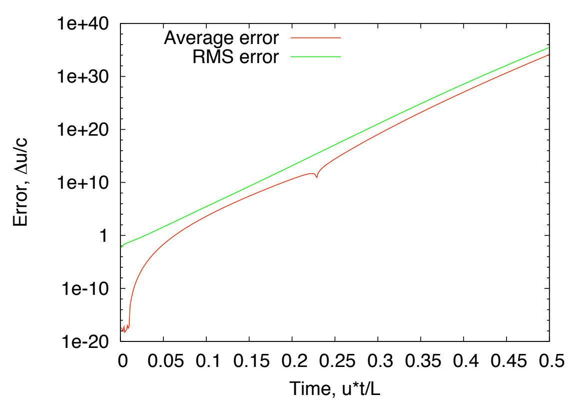


Figure 3: First order downwind in space, explicit Euler in time. Nx=100, CFL=0.1.