3.2 forward difference $\frac{\vec{\delta} f}{\delta x} = \frac{f_{i+1} - f_i}{\Delta x}$ (3.26)

central difference $\frac{\overleftrightarrow{\delta} f}{\delta x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$ (3.28)

3 pt forward difference $\frac{{}^{3}\vec{\delta} f}{\delta x} = \frac{-3f_i + 4f_{i+1} - f_{i+2}}{2\Delta x}$ (3.29)

function $f(x) = e^x \Rightarrow f'(x) = e^x = f(x)$

a) $x_i = 2.0$ $\Delta x = 0.1$ b) $x_i = 2.0$ $\Delta x = 0.2$

| | $f_{i-1}$ | $f_i$ | $f_{i+1}$ | $f_{i+2}$ |
|---|---|---|---|---|
| $\Delta x = 0.1$ | 6.6859 | 7.3891 | 8.1662 | 9.0250 |
| $\Delta x = 0.2$ | 6.0496 | 7.3891 | 9.0250 | 11.023 |

Compute the derivatives and errors

| | $\frac{\vec{\delta} f}{\delta x}$ | error | $\frac{\overleftrightarrow{\delta} f}{\delta x}$ | error | $\frac{{}^{3}\vec{\delta} f}{\delta x}$ | error |
|---|---|---|---|---|---|---|
| $\Delta x = 0.1$ | 7.7711 | 0.051709 | 7.4014 | $1.6675 \times 10^{-3}$ | 7.3625 | $-3.595 \times 10^{-3}$ |
| $\Delta x = 0.2$ | 8.1798 | 0.10701 | 7.4384 | $6.6800 \times 10^{-3}$ | 7.2743 | $-1.5534 \times 10^{-3}$ |
| $\frac{(\text{error})_{\Delta x = 0.1}}{(\text{error})_{\Delta x = 0.2}}$ | 0.4832 | | 0.24963 | | 0.23145 | |

$\simeq \frac{1}{2}$ $\simeq \frac{1}{2^2}$

3.3 The formula

$$\frac{u_{i+1j} + u_{i+1j-1} - u_{ij} - u_{ij-1}}{2\Delta x} + \frac{v_{i+1j} - v_{i+1j-1}}{\Delta y} = 0 \qquad (1)$$

Can be written as

$$\frac{u_{i+1j-\frac{1}{2}} - u_{ij-\frac{1}{2}}}{\Delta x} + \frac{v_{i+1j} - v_{i+1j-1}}{\Delta y} = 0 \qquad (2)$$

where

$$u_{ij-\frac{1}{2}} = \frac{1}{2}\left(u_{ij} + u_{ij-1}\right) \qquad (3)$$

Equation (2) can be written symbolically as

$$\frac{\vec{\delta}\, u_{ij-\frac{1}{2}}}{\delta x} + \frac{\overleftarrow{\delta}\, v_{i+1j}}{\delta y} = 0 \qquad (4)$$

which is in divergence form. Thus Eq. (1) is conservative.

3.7 verify

$$\frac{\partial^2 u}{\partial x^2} = \frac{-u_{i+3} + 4u_{i+2} - 5u_{i+1} + 2u_i}{h^2} + O(h^2) \qquad 3.40$$

Use Taylor series

$$-5\left[ u_{i+1} = u_i + u_{xi}h + \tfrac{1}{2}u_{xxi}h^2 + \tfrac{1}{6}u_{xxxi}h^3 + \tfrac{1}{24}u_{IVi}h^4 + \cdots \right]$$

$$4\left[ u_{i+2} = u_i + 2u_{xi}h + 2u_{xxi}h^2 + \tfrac{4}{3}u_{xxxi}h^3 + \tfrac{2}{3}u_{IVi}h^4 + \cdots \right]$$

$$-\left[ u_{i+3} = u_i + 3u_{xi}h + \tfrac{9}{2}u_{xxi}h^2 + \tfrac{9}{2}u_{xxxi}h^3 + \tfrac{27}{8}u_{IVi}h^4 + \cdots \right]$$

$$2\left[ u_i = u_i \right.$$

---

$$-u_{i+3} + 4u_{i+2} - 5u_{i+1} + 2u_i = \cancelto{0}{(-5 + 4 - 1 + 2)}u +$$

$$\cancelto{0}{(-5 + 8 - 3)}u_x h + \underbrace{\left(-\tfrac{5}{2} + 8 - \tfrac{9}{2}\right)}_{1}u_{xx}h^2 + \cancelto{0}{\left(-\tfrac{5}{6} + \tfrac{16}{3} - \tfrac{9}{2}\right)}u_{xxx}h^3 +$$

$$\underbrace{\left(-\tfrac{5}{24} + \tfrac{8}{3} - \tfrac{27}{8}\right)}_{1/3}u_{IV}h^4$$

solving for $u_{xx}$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{-u_{i+3} + 4u_{i+2} - 5u_{i+1} + 2u_i}{h^2} - \tfrac{1}{3}u_{IV}h^2}$$

3.8

$$-11\left[\; T_j = T_j \;\right]$$

$$18\left[\; T_{j+1} = T_j + T_j'\Delta y + \tfrac{1}{2}T_j''\Delta y^2 + \tfrac{1}{6}T_j'''\Delta y^3 + \tfrac{1}{24}T_j^{IV}\Delta y^4 \;\right]$$

$$-9\left[\; T_{j+2} = T_j + 2T_j'\Delta y + 2T_j''\Delta y^2 + \tfrac{4}{3}T_j'''\Delta y^3 + \tfrac{2}{3}T_j^{IV}\Delta y^4 \;\right]$$

$$2\left[\; T_{j+3} = T_j + 3T_j'\Delta y + \tfrac{9}{2}T_j''\Delta y^2 + \tfrac{9}{2}T_j'''\Delta y^3 + \tfrac{27}{8}T_j^{IV}\Delta y^4 \;\right]$$

---

$$\left(-11T_j + 18T_{j+1} - 9T_{j+2} + 2T_{j+3}\right) = \underbrace{\left(-11+18-9+2\right)}_{0}T_j \; +$$

$$\underbrace{\left(18 - 9(2) + 2(3)\right)}_{6}T_j'\Delta y \; + \; \underbrace{\left(\frac{18}{2} - 9(2) + 2\left(\frac{9}{2}\right)\right)}_{0}T''\Delta y^2 \; +$$

$$\underbrace{\left(\frac{18}{6} - 9\left(\frac{4}{3}\right) + 2\left(\frac{9}{2}\right)\right)}_{0}T_j'''\Delta y^3 \; + \; \underbrace{\left(\frac{18}{24} - 9\left(\frac{2}{3}\right) + 2\left(\frac{27}{8}\right)\right)}_{3/2}T_j^{IV}\Delta y^4$$

$$\frac{-11T_j + 18T_{j+1} - 9T_{j+2} + 2T_{j+3}}{6\Delta y} = T_j' + \frac{1}{4}T_j^{IV}\Delta y^3$$

or

$$T_j' = \frac{-11T_j + 18T_{j+1} - 9T_{j+2} + 2T_{j+3}}{6\Delta y} + O\left(\Delta y^3\right)$$

3.9 Verify

$$T_j = \frac{1}{11}\left[18\,T_{j+1} - 9\,T_{j+2} + 2\,T_{j+3} - 6h\,\frac{\partial T}{\partial y}\Big|_j\right] + O(h^4)$$

Use Taylor series

$$18\left[\,T_{j+1} = T_j + T_y h + \tfrac{1}{2}T_{yy}h^2 + \tfrac{1}{6}T_{yyy}h^3 + \tfrac{1}{24}T_{iv}h^4 + \cdots\right]$$

$$-9\left[\,T_{j+2} = T_j + 2T_y h + 2T_{yy}h^2 + \tfrac{4}{3}T_{yyy}h^3 + \tfrac{2}{3}T_{iv}h^4 + \cdots\right]$$

$$2\left[\,T_{j+3} = T_j + 3T_y h + \tfrac{9}{2}T_{yy}h^2 + \tfrac{9}{2}T_{xyy}h^3 + \tfrac{27}{8}T_{iv}h^4 + \cdots\right]$$

$$-6h\left[\,\frac{\partial T}{\partial y} = T_y\right]$$

---

$$18\,T_{j+1} - 9\,T_{j+2} + 2T_{j+3} - 6h\frac{\partial T}{\partial y} = \underbrace{(18-9+2)}_{11}T_j \;+$$

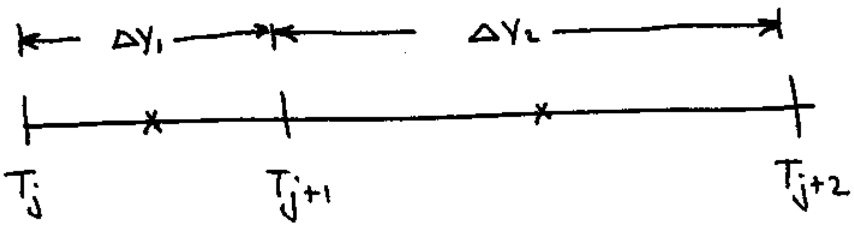
$$(18\;\overset{0}{\cancel{-18+6}}\;-6)\,T_y h \;+\; (9\;\overset{0}{\cancel{-18}}+9)\,T_{yy}h^2 \;+$$

$$(3\;\overset{0}{\cancel{-12}}+9)\,T_{xyy}h^3 \;+\; \underbrace{\left(\tfrac{3}{4} - 6 + \tfrac{27}{4}\right)}_{3/2}T_{iv}h^4 + \cdots$$

solving for $T_j$

$$\boxed{T_j = \frac{1}{11}\left[18\,T_{j+1} - 9T_{j+2} + 2T_{j+3} - 6h\frac{\partial T}{\partial y}\Big|_j\right] - \frac{3}{2}\,T_{iv}\,h^4}$$

3.15 Consider a non-uniform mesh




The derivatives at the interval midpoints are second order accurate

$$\left.\frac{\partial T}{\partial y}\right|_{j+\frac{1}{2}} \simeq \frac{T_{j+1} - T_j}{\Delta y_1} \equiv \left(\frac{\delta T}{\delta y}\right)_1$$

$$\left.\frac{\partial T}{\partial y}\right|_{j+3/2} \simeq \frac{T_{j+2} - T_{j+1}}{\Delta y^2} \equiv \left(\frac{\delta T}{\delta y}\right)_2$$

Now use linear extrapolation to estimate the derivative to second order at j

$$\left(\frac{\delta T}{\delta y}\right)_j = \left(\frac{\delta T}{\delta y}\right)_1 - \left[\left(\frac{\delta T}{\delta y}\right)_2 - \left(\frac{\delta T}{\delta y}\right)_1\right] \frac{\frac{1}{2}\Delta y_1}{\frac{1}{2}(\Delta y_1 + \Delta y_2)}$$

or

$$\left(\frac{\delta T}{\delta y}\right)_j = \left(\frac{2\Delta y_1 + \Delta y_2}{\Delta y_1 + \Delta y_2}\right)\left(\frac{\delta T}{\delta y}\right)_1 - \left(\frac{\Delta y_1}{\Delta y_1 + \Delta y_2}\right)\left(\frac{\delta T}{\delta y}\right)_2$$

$$\boxed{\left(\frac{\delta T}{\delta y}\right)_j = \frac{-(2\Delta y_1 + \Delta y_2)T_j + (3\Delta y_1 + \Delta y_2)T_{j+1} - \Delta y_1 T_{j+2}}{\Delta y_1 + \Delta y_2}}$$

3.16

$$\frac{\partial u}{\partial x} \simeq \frac{u_{i+1} - \alpha^2 u_{i-1} - (1-\alpha^2) u_i}{\Delta x^- \alpha(\alpha+1)} \qquad \alpha \equiv \frac{\Delta x^+}{\Delta x^-}$$

Use Taylor series

$$-\alpha^2 \left[ u_{i-1} = u_i - u_x \Delta x^- + \tfrac{1}{2} u_{xx} (\Delta x^-)^2 - \tfrac{1}{6} u_{xxx} (\Delta x^-)^3 + \dots \right]$$

$$\left[ u_{i+1} = u_i + u_x (\alpha \Delta x^-) + \tfrac{1}{2} u_{xx} (\alpha \Delta x^-)^2 + \tfrac{1}{6} u_{xxx} (\alpha \Delta x^-)^3 + \dots \right]$$

$$-(1-\alpha^2) \left[ u_i = u_i \right]$$

---

$$u_{i+1} - \alpha^2 u_{i-1} - (1-\alpha^2) u_i = \left[\cancelto{0}{\alpha^2 + 1 - (1 - \alpha^2)}\right] u_i +$$

$$(\alpha^2 + \alpha)(\Delta x^-) u_x + \left(\cancelto{0}{-\tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha^2}\right)(\Delta x^-)^2 u_{xx} +$$

$$\left( \frac{\alpha^2}{6} + \frac{\alpha^3}{6} \right)(\Delta x^-)^3 u_{xxx} + \dots$$

Solve for $u_x$

$$\boxed{\frac{\partial u}{\partial x} = \frac{u_{i+1} - \alpha^2 u_{i-1} - (1-\alpha^2) u_i}{\alpha(\alpha+1)\Delta x^-} - \frac{1}{6} u_{xxx} (\Delta x^-)^2 \alpha}$$

3.17 $\left.\frac{\partial T}{\partial y}\right|_1 = 0$

a) use a linear approximation

$$T = a + by + O(y^2)$$

now fit the data

$$\left.\begin{aligned} T_1 &= a + b(0) \\ T_2 &= a + b\Delta y \end{aligned}\right] \Rightarrow a = T_1 \quad b = \frac{T_2 - T_1}{\Delta y}$$

$$\left.\frac{dT}{dy}\right|_{y=0} = b = \frac{T_2 - T_1}{\Delta y} = 0 \Rightarrow \boxed{T_1 = T_2 \quad \text{order } \Delta y^2}$$

b) $T = a + by + cy^2 + O(y^3)$

fit the data

$$\left.\begin{aligned} T_1 &= a \\ T_2 &= a + b\Delta y + c\Delta y^2 \\ T_3 &= a + 2b\Delta y + 4c\Delta y^2 \end{aligned}\right\} \begin{aligned} a &= T_1 \\ b &= (-3T_1 + 4T_2 - T_3)/2\Delta y \\ c &= (T_1 - 2T_2 + T_3)/2\Delta y^2 \end{aligned}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = b = \frac{-3T_1 + 4T_2 - T_3}{2\Delta y} = 0 \Rightarrow \boxed{\begin{gathered} T_1 = \tfrac{1}{3}(4T_2 - T_3) \\ \text{order } \Delta y^3 \end{gathered}}$$

c)

$$T = a + by + cy^2 + dy^3 + O(y^3)$$

fit the data

$$T_1 = a$$

$$T_2 = a + b\Delta y + c\Delta y^2 + d\Delta y^3$$

$$T_3 = a + 2b\Delta y + 4c\Delta y^2 + 8d\Delta y^3$$

$$T_4 = a + 3b\Delta y + 9c\Delta y^2 + 27d\Delta y^3$$

write as a matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a \\ b\Delta y \\ c\Delta y^2 \\ d\Delta y^3 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$

Use Cramer's rule to find $b\Delta y$

$$\det = 12$$

determinate with rhs. inserted in Column 2

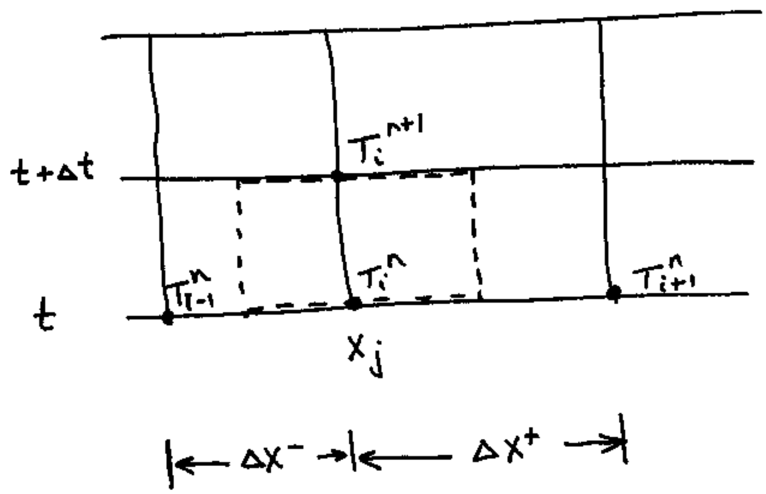
$$(\det)_2 = -22T_1 + 36T_2 - 18T_3 + 4T_4$$

$$b = \frac{-11T_1 + 18T_2 - 9T_3 + 2T_4}{6\Delta y}$$

$$\left.\frac{dT}{dy}\right|_{y=0} = b = \frac{-11T_1 + 18T_2 - 9T_3 + 2T_4}{6\Delta y} = 0$$

$$\boxed{T_1 = \frac{18T_2 - 9T_3 + 2T_4}{11} \quad \text{order } \Delta y^4}$$

3.19



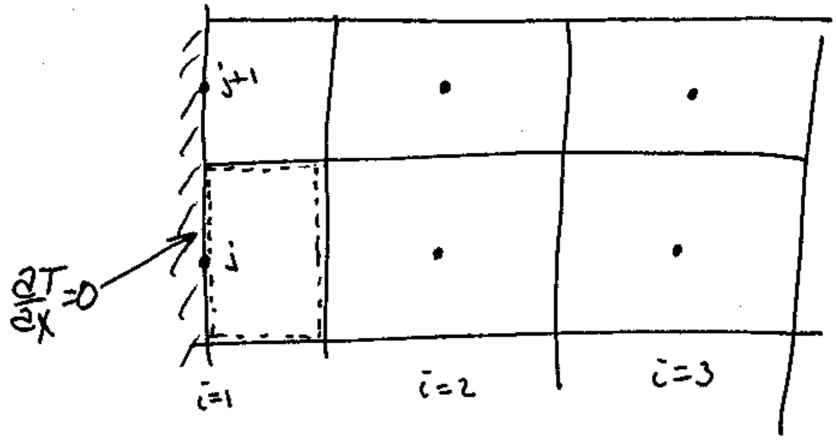

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} = \alpha \frac{\partial}{\partial x} \underbrace{\left(\frac{\partial T}{\partial x}\right)}_{\text{flux}}$$

evaluate the flux $\frac{\partial T}{\partial x}$ on the control surface

$$\left.\frac{\partial T}{\partial x}\right)_{i-\frac{1}{2}} = \frac{T_i^n - T_{i-1}^n}{\Delta x^-} \qquad \left.\frac{\partial T}{\partial x}\right)_{i+\frac{1}{2}} = \frac{T_{i+1} - T_i}{\Delta x^+}$$

$$\boxed{\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha}{\frac{1}{2}(\Delta x^+ + \Delta x^-)} \left[ \frac{T_{i+1} - T_i}{\Delta x^+} - \frac{T_i - T_{i-1}}{\Delta x^-} \right]}$$

3.20




$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

or in divergence form

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = 0$$

for cells i=1:

$$\left[\frac{T_{2j}-T_{1j}}{\Delta x} - 0\right]\Delta y + \left[\frac{T_{1j+1}-T_{1j}}{\Delta y} - \frac{T_{1j}-T_{1j-1}}{\Delta y}\right]\frac{\Delta x}{2} = 0$$

$$\boxed{2\left(\frac{T_{2j}-T_{1j}}{\Delta x^2}\right) + \frac{T_{1j+1} - 2T_{1j} + T_{1j-1}}{\Delta y^2} = 0}$$

1. (a) The fourth order central difference is

$$\frac{du}{dx} = \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + 2u_{i-2}}{12\Delta x} \qquad (1)$$

We determine the modified wavenumber by applying the above formula to a harmonic solution component

$$u_i = \hat{u}\, e^{ikx_i}$$

we find

$$u_{i+1} = e^{ik\Delta x}\hat{u} e^{ikx_i} = e^{ik\Delta x} u_i \qquad u_{i-1} = e^{-ik\Delta x} u_i$$

$$u_{i+2} = e^{i2k\Delta x} u_i \qquad u_{i-2} = e^{-i2k\Delta x} u_i$$

Substituting these forms into Eq. (1) yields

$$\frac{du}{dx} = \left( \frac{-e^{i2k\Delta x} + 8e^{ik\Delta x} - 8e^{-ik\Delta x} + e^{i2k\Delta x}}{12\Delta x} \right) u_i$$

$$= \left[ \frac{4}{3}\left( \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2} \right) - \frac{1}{6}\left( \frac{e^{ik2\Delta x} - e^{-ik2\Delta x}}{2} \right) \right] \frac{u_i}{\Delta x}$$

$$= \left[ \frac{4}{3} \sinh(ik\Delta x) - \frac{1}{6} \sinh(i2k\Delta x) \right] \frac{u_i}{\Delta x}$$

$$= i \underbrace{\left[ \frac{8\sin(k\Delta x) - \sin(2k\Delta x)}{6\Delta x} \right]}_{k_{eff_4}} \hat{u}\, e^{ikx_i} \qquad (2)$$

For a second order central difference, the effective wavenumber is given in the text as

$$k_{eff_2} = \frac{\sin k\Delta x}{\Delta x} \qquad (3)$$

Define non-dimensional errors as follows

$$\frac{E_4 \Delta x_0}{\pi} = \left(\frac{K\Delta x}{\pi} - \frac{K_{eff_4}\Delta x}{\pi}\right)\frac{\Delta x_0}{\Delta x} = \left[\frac{K\Delta x}{\pi} - \frac{8\sin K\Delta x - \sin 2K\Delta x}{6\pi}\right]\frac{\Delta x_0}{\Delta x}$$

$$\frac{E_2 \Delta x_0}{\pi} = \left(\frac{K\Delta x}{\pi} - \frac{K_{eff_2}\Delta x}{\pi}\right)\frac{\Delta x_0}{\Delta x} = \left[\frac{K\Delta x}{\pi} - \frac{\sin K\Delta x}{\pi}\right]\frac{\Delta x_0}{\Delta x}$$

where $\Delta x_0$ is a reference mesh spacing. Now define

$$\xi \equiv \frac{K\Delta x}{\pi} \qquad r \equiv \frac{\Delta x_0}{\Delta x}$$

so that the error expressions become

$$\frac{E_4 \Delta x_0}{\pi} = r\left[\xi - \left(\frac{8\sin\pi\xi - \sin 2\pi\xi}{6\pi}\right)\right]$$

$$\frac{E_2 \Delta x_0}{\pi} = r\left[\xi - \frac{\sin\pi\xi}{\pi}\right]$$

Note that $\xi$ ranges from 0 to 1 for any mesh. If we plot the non-dimensional error as a function of $K\Delta x_0/\pi = r\xi$ we see that the mesh refinement factor $r$ simply stretches the coordinate $\xi$ by $r$, while simultaneously magnifying the error by the same factor. The end result is a reduction in error over the wavenumber range associated with the base mesh. The error is also smaller than that for the $4^{th}$ order scheme, except for the lowest wavenumbers. Thus mesh refinement with the $2^{nd}$ order scheme is slightly less accurate than the $4^{th}$ order scheme at low wavenumbers, and more accurate at high wavenumbers. The following plots illustrate this effect.

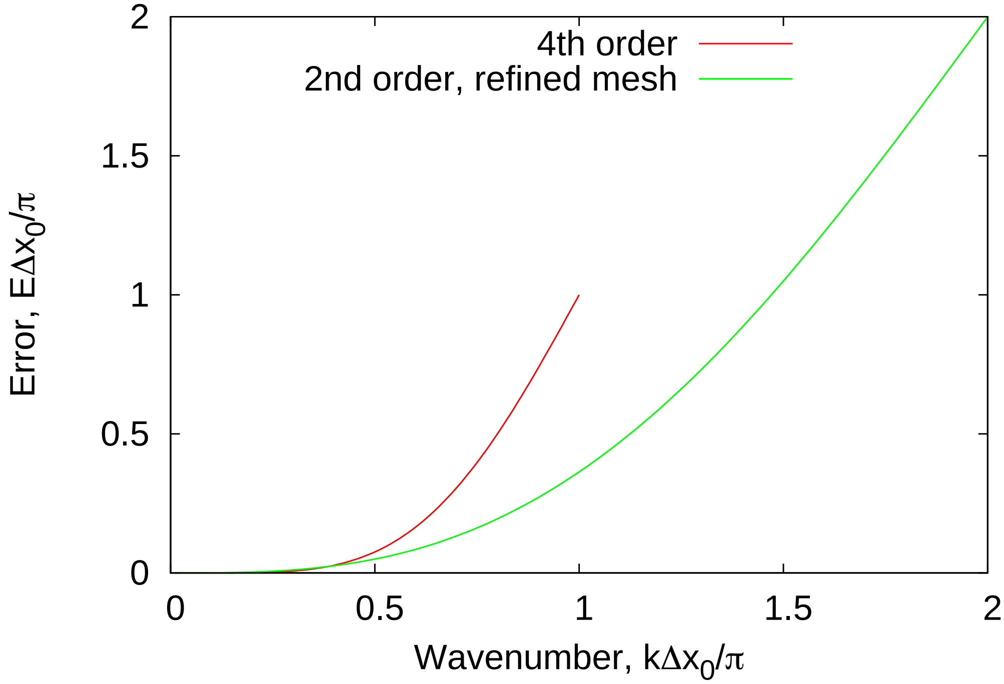



Figure 1: Error distributions for a 4th order scheme compared with a second order scheme on a mesh refined by a factor of 2. Note that the error for the refined 2nd order scheme is lower than the error for the 4th order scheme over much of the wavenumber range associated with the base mesh.

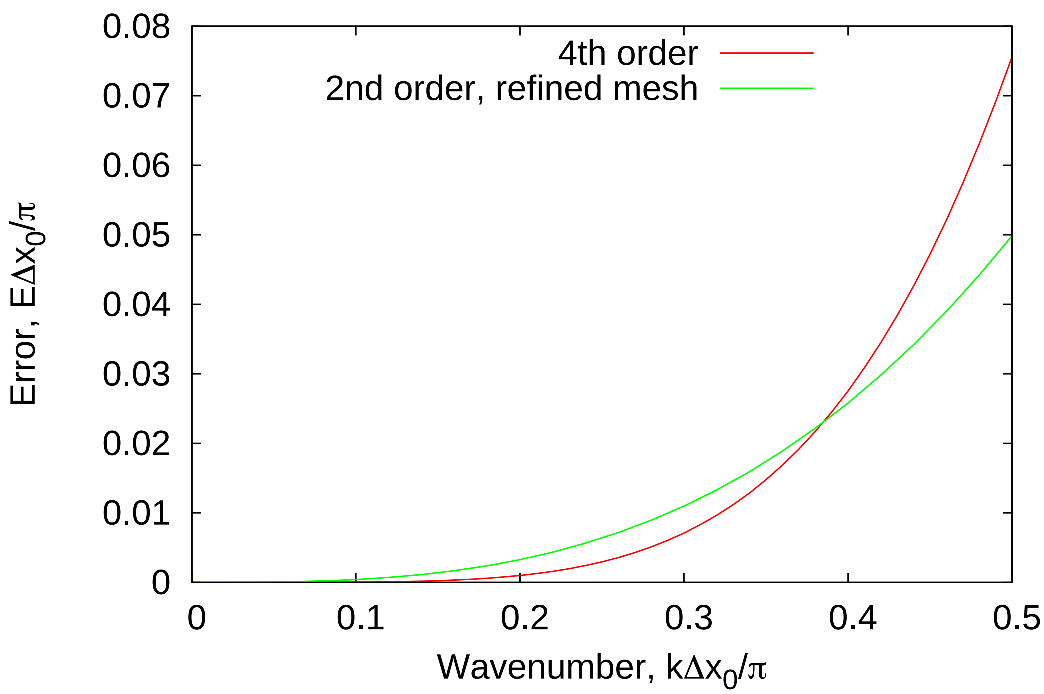



Figure 2: Zoom of the previous figure for low wavenumbers. Here we see that the 4th order scheme is slightly more accurate than the refined 2nd order scheme for low wavenumbers.

2. (a) The advection-diffusion equation is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) \tag{1}$$

where $c$ and $\nu$ are constants. Assume a domain $0 \le x \le L$ with boundary conditions $u(x=0,t) = 0$ and $u(x=L,t) = u_1$. Introduce non-dimensional variables

$$x^* = \frac{x}{L}, \quad t^* = \frac{ct}{L}, \quad u^* = \frac{u}{c}$$

So that Eq. (1) can be written as

$$\frac{\partial(cu^*)}{\partial(Lt^*/c)} + c\frac{\partial(cu^*)}{\partial(Lx^*)} = \frac{\partial}{\partial(Lx^*)}\left(\nu\frac{\partial(cu^*)}{\partial(Lx^*)}\right)$$

which simplifies to

$$\frac{\partial u^*}{\partial t^*} + \frac{\partial u^*}{\partial x} = \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} \tag{2}$$

where the Reynolds number is defined as

$$Re = \frac{cL}{\nu} \tag{3}$$

b) The steady-state solution is

$$u^*_\infty(x^*) = u_1^*\left[\frac{\exp(Re\,x^*) - 1}{\exp(Re) - 1}\right] \tag{4}$$

To verify that this solution is correct, substitute it into Eq. (2) to get

$$0 + \frac{u_1^* Re \exp(Re\,x^*)}{\exp(Re) - 1} = \frac{1}{\cancel{Re}}\left[\frac{u_1^* Re^{\cancel{2}} \exp(Re\,x^*)}{\exp(Re) - 1}\right]$$

which is an equality. We also observe $U_\infty^*(x^*=0) = 0$ and $U_\infty^*(x^*=1) = U_1^*$ in accordance with the boundary conditions. Thus Eq. (4) is indeed the steady-state solution to Eq. (2)

(c) Assume that an initial condition $U_0^*(x^*)$ is specified. We can find the transient solution by proceeding as follows. First introduce the new variable $w$, defined as

$$W = U(x,t) - U_\infty(x) \tag{5}$$

So that the boundary conditions become homogeneous

$$w(0,t) = 0 \qquad w(1,t) = 0 \qquad w(x,0) = U_0(x) - U_\infty(x)$$

Here and in what follows we have dropped the $*$, with the understanding that all variables are non-dimensional. The transformation (5) leaves the form of the advection-diffusion equation unchanged, viz

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = \alpha \frac{\partial^2 w}{\partial x^2} \tag{6}$$

where $\alpha = 1/Re$. We now introduce the Cole-Hopf transformation in order to map the above equation into a pure diffusion equation. The transformation is

$$W(x,t) = \underbrace{\exp\left(\frac{x}{2\alpha} - \frac{t}{4\alpha}\right)}_{E} v(x,t) \tag{7}$$

Substitution of this form into Eq. (6) yields

$$\frac{-1}{4\alpha} E v + E \dot{v} + \frac{1}{2\alpha} E v + E v' =$$

$$\alpha\left( \frac{1}{4\alpha^2} E v + \frac{1}{2\alpha} E v' + \frac{1}{2\alpha} E v' + E v'' \right)$$

$$\cancel{\frac{1}{4\alpha} E v} + E \dot{v} + \cancel{E v'} = \cancel{\frac{1}{4\alpha} E v} + \cancel{E v'} + \alpha E v''$$

$$\dot{v} = \alpha v'' \qquad (8)$$

The boundary and initial conditions map as follows

$$v(0,t) = 0 \qquad v(1,t) = 0 \qquad v(x,0) = \exp\left(\frac{-x}{2\alpha}\right)\left(U_0(x) - U_\infty(x)\right) \qquad (9)$$

The solution to Eq. (8) is

$$v(x,t) = \sum_{n=1}^{\infty} A_n \exp(-\alpha \pi^2 n^2 t) \sin(n\pi x) \qquad (10)$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} v_0(x) \sin(n\pi x)\, dx \qquad (11)$$

Now combining Eqs (5), (6), (9), (10) and (11), the solution can be written as

$$U(x,t) = \exp\left(\frac{x}{2\alpha} - \frac{t}{4\alpha}\right)\sum_{n=1}^{\infty} A_n \exp(-\alpha n^2 \pi^2 t) \sin(n\pi x) + U_\infty(x) \qquad (12)$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \exp\left(\frac{-x}{2\alpha}\right)\left(U_0(x) - U_\infty(x)\right) \sin(n\pi x) \qquad (13)$$

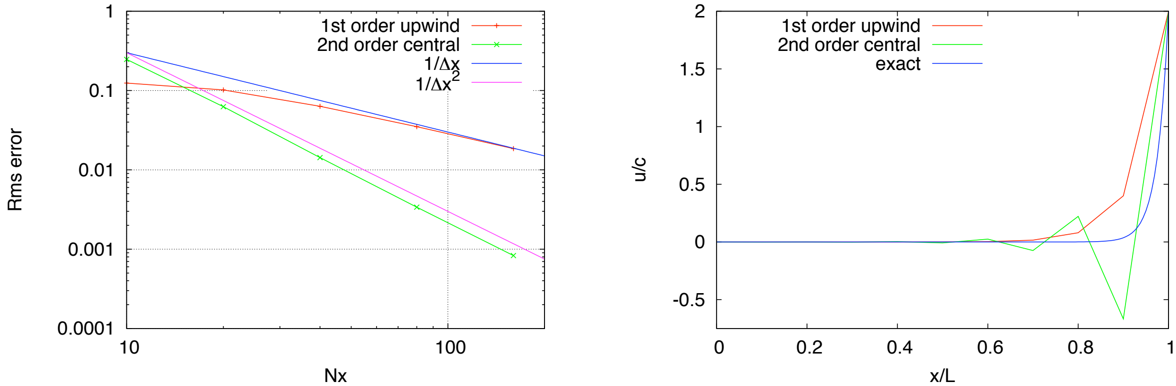



Figure 1: Left - Convergence to the advection-diffusion equation on a uniform mesh. Re=40. The orders of accuracy are confirmed to be first order for the upwind approximation and second order for the central difference. Right - Solutions at $t^* = 5$ for Nx=10.

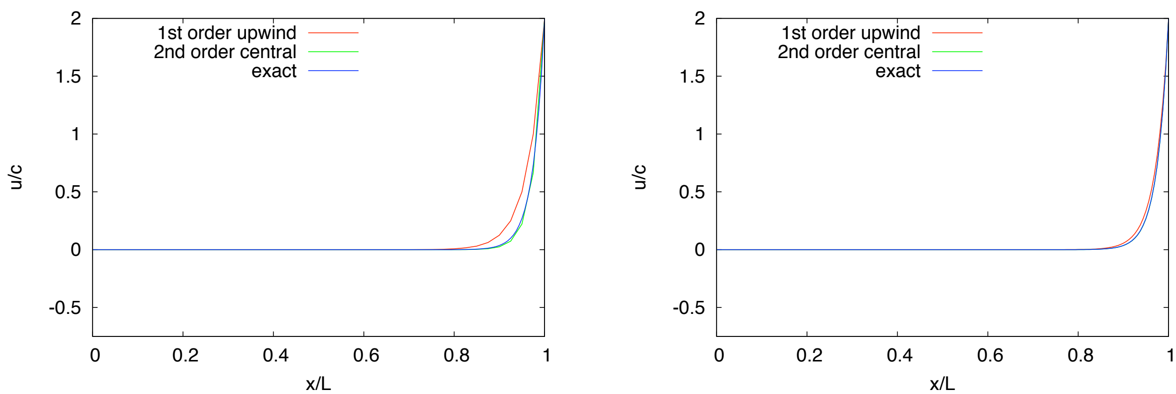



Figure 2: Left - Solutions at $t^* = 5$ for Nx=40, Re=40. Right - Solutions at $t^* = 5$ for Nx=160, Re=40.

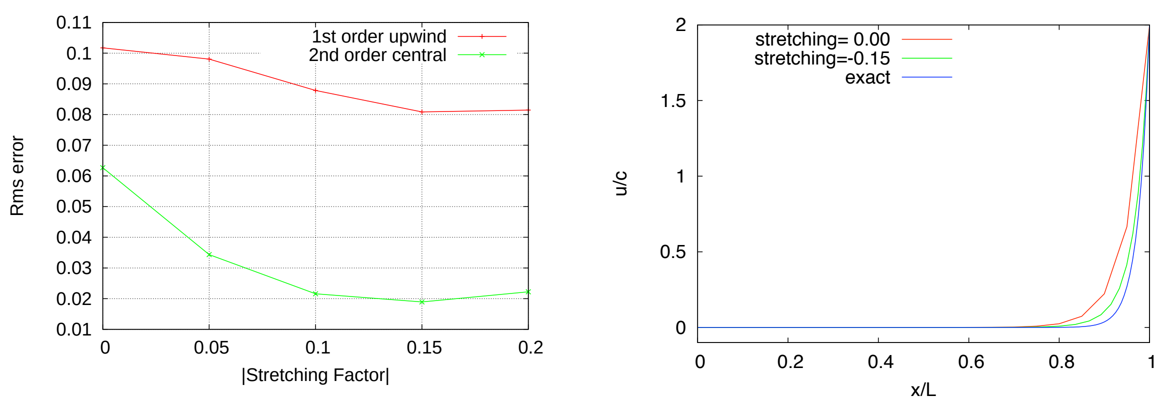



Figure 3: Left - Convergence to the advection-diffusion equation on a stretched mesh. Nx=20, Re=40. Note that a stretching factor of about $\sigma = -0.15$ is optimal in this case since further clustering places too many points near the right boundary and too few points over the remainder of the domain. Right - Solutions at $t^* = 5$ for the first order upwind scheme. Nx=20.

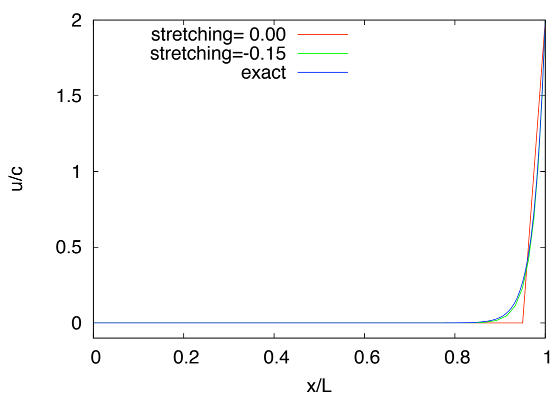



Figure 4: Solutions at $t^* = 5$ for the second order central difference scheme. Nx=20.