ASEN 5327 Computational Fluid Dynamics
Spring 2009

Homework 3 Solution

It is useful to develop a few frequently used relations prior to working out amplification factors for various numerical schemes. Under a Fourier decomposition, an individual solution component to a marching problem has the form

$$u_m(x,t) = u_0(m)e^{\lambda_m t}e^{ik_m x} \tag{1}$$

On an equispaced mesh characterized by Δx and Δt, we can use the shorthand notations $x_j = j\Delta x$, $t = n\Delta t$, $(u_m)_j^n = u_m(x_j,t)$ to write Eq. (1) as

$$(u_m)_j^n = u_0(m)e^{\lambda_m t}e^{ik_m x_j} \tag{2}$$

Using the above form, $(u_m)_j^{n+1}$ can be written as

$$\begin{aligned}(u_m)_j^{n+1} &= u_0(m)e^{\lambda_m (t+\Delta t)}e^{ik_m x_j} \\ &= u_0(m)e^{\lambda_m t}e^{ik_m x_j}e^{\lambda_m \Delta t} \\ &= (u_m)_j^n G \end{aligned} \tag{3}$$

where the amplification factor, G is defined as

$$G = e^{\lambda_m \Delta t} \tag{4}$$

Similarly

$$(u_m)_j^{n-1} = (u_m)_j^n \left(\frac{1}{G}\right) \tag{5}$$

Shifts in space result in similar expressions

$$(u_m)_{j+1}^n = (u_m)_j^n e^{ik_m \Delta x} = (u_m)_j^n e^{i\beta} \tag{6}$$

$$(u_m)_{j-1}^n = (u_m)_j^n e^{-ik_m \Delta x} = (u_m)_j^n e^{-i\beta} \tag{7}$$

where the definition of the non-dimensional wavenumber, $\beta = k_m \Delta x$ is apparent.

Using the above results, the transforms of several common difference operators are given below

$$\begin{aligned}\frac{\vec{\delta}\, u}{\delta x} = \frac{u_{j+1}^n - u_j^n}{\Delta x} &= \frac{(u_m)_j^n}{\Delta x}\left(e^{ik_m\Delta x} - 1\right) \\ &= \frac{(u_m)_j^n}{\Delta x}\left((\cos\beta - 1) + i\sin\beta\right) \\ &= \frac{(u_m)_j^n}{\Delta x}\left[-2\sin^2\left(\frac{\beta}{2}\right) + i\sin\beta\right]\end{aligned} \tag{8}$$

$$\frac{\overleftarrow{\delta}\, u}{\delta x} = \frac{u_j^n - u_{j-1}^n}{\Delta x} = \frac{(u_m)_j^n}{\Delta x}\left[2\sin^2\left(\frac{\beta}{2}\right) + i\sin\beta\right] \tag{9}$$

$$\frac{\overleftrightarrow{\delta}\, u}{\delta x} = \frac{1}{2}\left(\frac{\overrightarrow{\delta}\, u}{\delta x} + \frac{\overleftarrow{\delta}\, u}{\delta x}\right) = \frac{(u_m)_j^n}{\Delta x}\left[i\sin\beta\right] \tag{10}$$

$$\frac{\overleftrightarrow{\delta}^2 u}{\delta x^2} = \frac{1}{\Delta x^2}\left(\frac{\overrightarrow{\delta}\, u}{\delta x} - \frac{\overleftarrow{\delta}\, u}{\delta x}\right) = -\frac{(u_m)_j^n}{\Delta x}\left[4\sin^2\left(\frac{\beta}{2}\right)\right] \tag{11}$$

3.25 The scheme

$$u_j^{n+1} = u_j^n - \underbrace{\frac{c\Delta t}{\Delta x}}_{\nu}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2}\right) \tag{12}$$

can be written as

$$u_j^{n+1} = u_j^n - \nu\Delta x\frac{\overleftrightarrow{\delta}\, u_j^n}{\delta x} \tag{13}$$

Making use of Eqs. (2), (3), (10), the above equation is transformed to Fourier space.

$$(u_m)_j^n G = (u_m)_j^n - \nu(u_m)_j^n i\sin\beta \tag{14}$$

The term common $(u_m)_j^n$ term cancels, allowing the above expression to be solved for the amplification factor

$$G = 1 - i\nu\sin\beta \tag{15}$$

Its modulus is simply

$$|G|^2 = 1 + \nu^2\sin^2\beta \tag{16}$$

Which is greater than 1 for any value of ν and thus the explicit scheme is unconditionally unstable.

If the spatial derivative in Eq. 13 is evaluated at time level $n+1$, an extra factor of G will appear on the right hand side at the point of Eq. (14)

$$(u_m)_j^n G = \left[(u_m)_j^n - \nu(u_m)_j^n i\sin\beta\right] G \tag{17}$$

Solving this equation for G we have

$$G = \frac{1}{1 - i\nu\sin\beta} \tag{18}$$

Its modulus is simply

$$|G|^2 = \frac{1}{1 + \nu^2\sin^2\beta} \tag{19}$$

which is less than 1 for all ν and thus the implicit scheme is unconditionally stable.

3.26 The Du Fort Frankel scheme is

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{\Delta x^2}\left(u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n\right) \tag{20}$$

Making use of Eqs. (2), (3), (5), (6), (7), and cancelling the common $(u_m)_j^n$ term leads to

$$G + \frac{1}{G} = 2r\left(e^{i\beta} - G - \frac{1}{G} + e^{-i\beta}\right) \tag{21}$$

where $r = (\alpha\Delta t/\Delta x^2)$. The exponential terms combine to give $2\cos\beta$. Making use of this result, as well as multiplying the above equation through by G and simplifying leads to

$$(2r+1)G^2 - 4r\cos\beta G + (2r-1) = 0 \tag{22}$$

Solving the quadratic, we have

$$G = \frac{2r\cos\beta \pm \sqrt{1 - 4r^2\sin^2\beta}}{2r+1} \tag{23}$$

Consider the amplification factor for large r. In this case, the above relation can be approximated as

$$G \simeq \frac{2r\cos\beta \pm i2r\sin\beta}{2r} = \cos\beta \pm i\sin\beta \tag{24}$$

In this limit, the amplification factor is seen to be of modulus 1. Thus r can be taken arbitrarily large and the scheme is unconditionally stable.

3.28 The following scheme

$$u_j^{n+1} = u_j^n + \frac{\alpha\Delta t}{\Delta x^2}\left[\frac{1}{3}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) + \frac{2}{3}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\right] \tag{25}$$

can be written as

$$u_j^{n+1} = u_j^n + r\Delta x^2\left[\frac{1}{3}\frac{\overset{\leftrightarrow}{\delta}{}^2 u_j^{n+1}}{\delta x^2} + \frac{2}{3}\frac{\overset{\leftrightarrow}{\delta}{}^2 u_j^n}{\delta x^2}\right] \tag{26}$$

Making use of Eqs. (2), (3), (11), and cancelling the common $(u_m)_j^n$ term leads to

$$G = 1 - \frac{4r}{3}(G+2)\sin^2\left(\frac{\beta}{2}\right) \tag{27}$$

Solving for G we have

$$G = \frac{3 - 8r\sin^2\left(\frac{\beta}{2}\right)}{3 + 4r\sin^2\left(\frac{\beta}{2}\right)} \tag{28}$$

Since the numerator decreases with β and the denominator increases with β, there is danger that the amplification factor drops below -1. The most restrictive wavenumber is $\beta = \pi$. In this limit, the stability condition reads

$$G = \frac{3 - 8r}{3 + 4r} \geq -1 \tag{29}$$

which leads to

$$r \leq \frac{3}{2} \tag{30}$$

3.29 Consider the implicit scheme

$$u_j^{n+1} = u_j^n - \underbrace{\frac{c\Delta t}{\Delta x}}_{\nu} \left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} \right) \tag{31}$$

which can be written as

$$u_j^{n+1} = u_j^n - \nu \Delta x \frac{\overrightarrow{\delta}\, u_j^{n+1}}{\delta x} \tag{32}$$

Making use of Eqs. (2), (3), (8), and cancelling the common $(u_m)_j^n$ term leads to

$$G = 1 - \nu \left[-2\sin^2\left(\frac{\beta}{2}\right) + i\sin\beta \right] G \tag{33}$$

Solving for the amplification factor, we have

$$G = \frac{1}{\left[1 - 2\nu\sin^2\left(\frac{\beta}{2}\right)\right] + i\nu\sin\beta} \tag{34}$$

Its modulus is then computed to be

$$|G|^2 = \frac{1}{1 + 4\nu(\nu - 1)\sin^2\left(\frac{\beta}{2}\right)} \tag{35}$$

The denominator will be less than 1 only if the factor $4\nu(\nu - 1)$ is less than 0. This condition occurs when $0 \leq \nu \leq 1$. Thus the scheme is stable everywhere outside this range, or

$$\nu \leq 0, \qquad \nu \geq 1 \tag{36}$$

3.30 The leap frog scheme applied to the wave equation can be written as

$$u_j^{n+1} = u_j^{n-1} - 2\nu\Delta x \frac{\overleftrightarrow{\delta}\, u_j^n}{\delta x} \tag{37}$$

Making use of Eqs. (2), (3), (5), (10), and cancelling the common $(u_m)_j^n$ term leads to

$$G = \frac{1}{G} - i2\nu\sin\beta \tag{38}$$

Multiplying through by G and rearranging, we have

$$G^2 + i2\nu\sin\beta G - 1 = 0 \tag{39}$$

Solving for G

$$G = -i\nu \sin\beta \pm \sqrt{1 - \nu^2 \sin^2\beta} \tag{40}$$

If $\nu^2 \sin^2\beta \leq 1$, the argument of the square root above is positive and G is a complex number. In this case the modulus is simply

$$|G|^2 = \nu^2 \sin^2\beta + (1 - \nu^2 \sin^2\beta) = 1 \tag{41}$$

which satisfies the stability criterion. The condition $\nu^2 \sin^2\beta \leq 1$ will be met for all β if $\nu^2 \leq 1$.

If $\nu^2 \sin^2\beta / > 1$, the amplification factor is a pure imaginary number and can be written as

$$G = -i\left(\nu \sin\beta \pm \sqrt{\nu^2 \sin^2\beta - 1}\right) \tag{42}$$

If $\nu^2 \sin^2\beta / > 1$, it is clear that the positive root will exceed 1 when $\nu > 0$, and the negative root will drop below -1 when $\nu < 0$. Thus the scheme is unstable for $\nu^2 > 1$ we have the stability bound

$$\nu^2 \leq 1 \tag{43}$$

3.34 The matrix

$$\begin{bmatrix} (1+\nu) & \nu & 0 \\ 0 & (1+\nu) & \nu \\ -\nu & 0 & (1+\nu) \end{bmatrix} \tag{44}$$

has the following characteristic equation

$$\begin{vmatrix} (1+\nu-\lambda) & \nu & 0 \\ 0 & (1+\nu-\lambda) & \nu \\ -\nu & 0 & (1+\nu-\lambda) \end{vmatrix} = 0 \tag{45}$$

or

$$(1+\nu-\lambda)^3 - \nu^3 = 0 \tag{46}$$

which leads to

$$(1+\nu-\lambda) = (1)^{\frac{1}{3}}\nu \tag{47}$$

Noting that $(1)^{\frac{1}{3}} = 1,\ \exp(i2\pi/3),\ \exp(i4\pi/3)$, the roots are

$$\lambda_1 = 1, \quad \lambda_2 = (1 + \frac{3\nu}{2}) + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = (1 + \frac{3\nu}{2}) - i\frac{\sqrt{3}}{2} \tag{48}$$

The modulus of the second and third eigenvalues is

$$|\lambda_{2,3}|^2 = 1 + 3\nu + 3\nu^2 \tag{49}$$

For stability we require $|\lambda_{max}| \le 1$. The only possible restriction will come from the modulus of $\lambda_{2,3}$ given above. This modulus exceeds 1 for positive ν and drops below -1 when $\nu = -1$. Thus the stability bound is

$$-1 \le \nu \le 0 \tag{50}$$

3.35 The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ r & (1-2r) & r \\ 0 & 0 & 1 \end{bmatrix} \tag{51}$$

has the following characteristic equation

$$(1-\lambda)^2(1-2r-\lambda) = 0 \tag{52}$$

The roots are

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1 - 2r \tag{53}$$

For stability we require $|\lambda_{max}| \le 1$. The third eigenvalue is the maximum, and thus the stability bound is

$$r \le 1 \tag{54}$$

3.36 The upstream scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \tag{55}$$

can be written as

$$u_j^{n+1} = (1-\nu)u_j^n + \nu u_{j-1}^n \tag{56}$$

with the boundary conditions,

$$u_1^n = 1 \qquad u_4^{n+1} = u_3^n, \tag{57}$$

the above difference scheme for a four-point mesh takes the form

$$\begin{Bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \nu & (1-\nu) & 0 & 0 \\ 0 & \nu & (1-\nu) & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1^n \\ u_2^n \\ u_3^n \\ u_4^n \end{Bmatrix} \tag{58}$$

Notice that the boundary condition implied by the first row in the matrix is actually $u_1^{n+1} = u_1^n$. This works since u_1 is initialized to 1 and thus the boundary condition simply insures that $u_1 = 1$ for all time.

The matrix has the following characteristic equation

$$-(1-\lambda)(1-\nu-\lambda)^2\lambda = 0 \tag{59}$$

which has the roots

$$\lambda_1 = 1, \quad \lambda_2 = 1-\nu, \quad \lambda_3 = 1-\nu, \quad \lambda_4 = 0 \tag{60}$$

For stability we require $|\lambda_{max}| \leq 1$. The second and third eigenvalues are maximum, and thus the stability bound is

$$0 \leq \nu \leq 2 \tag{61}$$

1.

(a) The advection-diffusion equation is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right), \tag{62}$$

which can also be written as

$$\frac{\partial u}{\partial t} = c\left[\left(\frac{\nu}{c}\right)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right]. \tag{63}$$

After approximating the spatial derivative with second order central differences we have

$$\frac{\partial u}{\partial t} = c\left[\left(\frac{\nu}{c}\right)\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) - \left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right)\right] \tag{64}$$

Taking the Fourier transform leads to

$$\frac{\partial \hat{u}}{\partial t} = c\left[\left(\frac{\nu}{c}\right)\left(\frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2}\right) - \left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}\right)\right]\hat{u}_i \tag{65}$$

or

$$\frac{\partial \hat{u}}{\partial t} = \underbrace{-\frac{c}{\Delta x}\left[\underbrace{\left(\frac{\nu}{c\Delta x}\right)}_{1/R_\Delta} 2[1 - \cos(k\Delta x)] \; + \; i\sin(k\Delta x)\right]}_{\lambda}\hat{u}_i. \tag{66}$$

The stability analysis requires $\lambda\Delta t$ which is

$$\lambda\Delta t = -\underbrace{\left(\frac{c\Delta t}{\Delta x}\right)}_{CFL}\left[\frac{2[1 - \cos(k\Delta x)]}{R_\Delta} \; + \; i\sin(k\Delta x)\right]. \tag{67}$$

If we set $CFL = R_\Delta/2$, the minimum real part of $\lambda\Delta t$ will be -2. Figure 1 shows the eigenvalue spectrum with this prescription for CFL, plotted for R_Δ=1.0, 2.0, 3.0. Included in this plot is the unit circle centered at $\lambda\Delta t = -1$, which is the stability bound for the explicit Euler method.

(b) Note that, with the CFL number scaled as indicated above, the eigenvalues fall on, or within the unit circle for all $R_\Delta \leq 2$. Thus we have the condition

$$CFL \leq \frac{R_\Delta}{2} \quad \text{for} \quad R_\Delta \leq 2 \tag{68}$$

(c) From Figure 1 we see that when $R_\Delta > 2$ the eigenvalues cross the explicit Euler stability bound near the origin. We can therefore undertake a series expansion of Eq. (67) near $k\Delta x = 0$ in order to determine the time step restriction in this case. Doing this results in

$$\lambda\Delta t \simeq -(CFL)\left[\frac{(k\Delta x)^2}{R_\Delta} \; + \; ik\Delta x\right]. \tag{69}$$

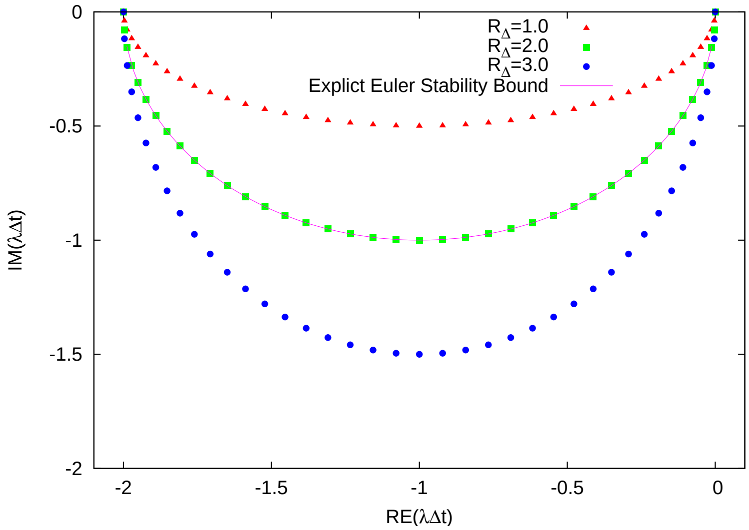

Figure 1: Eigenvalue spectra for second order central differences applied to the advection diffusion equation. The eigenvalues are scaled so that $(\lambda\Delta t)_{min} = -2$. The stability bound for the explicit Euler time advancement scheme is also shown.

The stability restriction for the explicit Euler method is $|1 + \lambda\Delta t|^2 \leq 1$, which can be applied to the above approximation to get

$$|1 + \lambda\Delta t| = |1 - (CFL)\left[\frac{(k\Delta x)^2}{R_\Delta} + ik\Delta x\right]|^2 \leq 1 \tag{70}$$

or upon collecting real and imaginary parts and forming the modulus squared,

$$1 - \frac{2CFL(k\Delta x)^2}{R_\Delta} + \frac{CFL^2(k\Delta x)^4}{R_\Delta^2} + CFL^2(k\Delta x)^2 \leq 1. \tag{71}$$

Neglecting the term of order $(k\Delta x)^4$ and simplifying leads to

$$CFL(k\Delta x)\left[CFL - \frac{2}{R_\Delta}\right] \leq 0, \tag{72}$$

which implies

$$CFL \leq \frac{2}{R_\Delta} \quad \text{for} \quad R_\Delta > 2. \tag{73}$$

(d) Since Eq. (68) is a strictly increasing function of R_Δ, and Eq. (73) is a strictly decreasing function, a unified expression may be formed simply by taking the minimum of the two, viz

$$CFL \leq \min\left[\frac{R_\Delta}{2}, \frac{2}{R_\Delta}\right]. \tag{74}$$

The inviscid limit ($\nu \to 0$) is achieved when $R_\Delta \to \infty$. The above results shows that the CFL number (and hence the time step) must approach zero in this limit. This is the expected result since one is basically solving the wave equation in the inviscid limit and a central difference in conjunction with the explicit Euler time advancement scheme is unconditionally unstable.

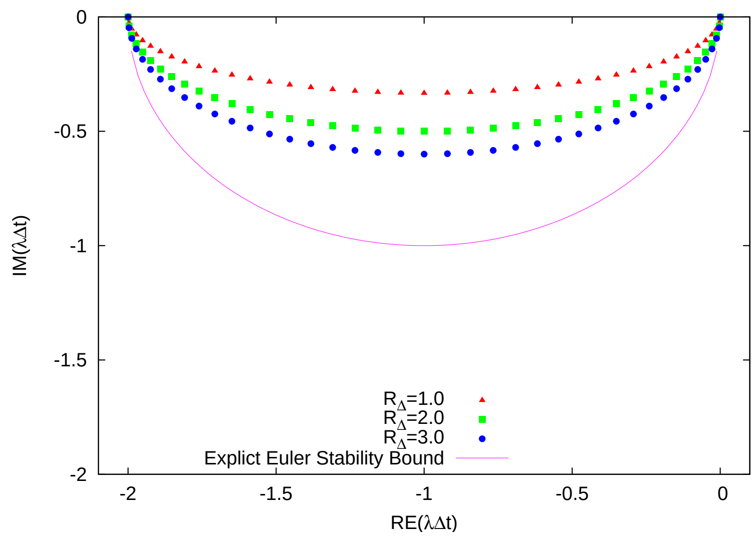

Figure 2: Eigenvalue spectra for first order upwind difference of the advective term, second order central difference of the diffusion term. The eigenvalues are scaled so that $(\lambda\Delta t)_{min} = -2$. The stability bound for the explicit Euler time advancement scheme is also shown.

2.

(a) With a first order upwind difference applied to the advection term, the discrete form of the advection-diffusion equation is

$$\frac{\partial u}{\partial t} = c\left[\left(\frac{\nu}{c}\right)\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) - \left(\frac{u_{i+1} - u_i}{\Delta x}\right)\right] \tag{75}$$

Which leads to the following expression for the eigenvalues

$$\lambda\Delta t = -CFL\left[\left(\frac{2}{R_\Delta} + 1\right)[1 - \cos(k\Delta x)] \ + \ i\sin(k\Delta x)\right]. \tag{76}$$

If we set $CFL = R_\Delta/(2+R_\Delta)$, the minimum real part of $\lambda\Delta t$ will be -2. Figure 2 shows the eigenvalue spectrum with this prescription for CFL, plotted for R_Δ=1.0, 2.0, 3.0. Included in this plot is the unit circle centered at $\lambda\Delta t = -1$, which is the stability bound for the explicit Euler method.

(b) From Figure 2, we see that the eigenvalues fall on or within the unit circle for all values of R_Δ chosen. A little more analysis shows that for $R_\Delta \to \infty$, the expression for $\lambda\Delta t$ becomes

$$\lambda\Delta t = -CFL\left[(1 - \cos(k\Delta x)) \ + \ i\sin(k\Delta x)\right]. \tag{77}$$

which is the equation for a unit circle centered at $\lambda_r\Delta t = -1$. Thus the eigenvalues fall on or within the circle for all values of R_Δ, provided

$$CFL \leq \frac{R_\Delta}{2 + R_\Delta}. \tag{78}$$

1st order upwind			2nd order central	
R_Δ	CFL, theory	CFL, measured	CFL, theory	CFL, measured
0.5	0.20	0.20309	0.25	0.25527
2.0	0.50	0.52876	1.00	1.09841
8.0	0.80	0.87160	0.25	0.37500

Table 1: Comparison of predicted and measured maximum CFL numbers for the advection-diffusion equation. 40 point uniform mesh and second order central difference applied to the diffusion term in all cases.

(c) The inviscid limit ($\nu \to 0$) is achieved when $R_\Delta \to \infty$. The above results shows that the CFL number approaches 1.0 in this limit. This is the expected result since one is basically solving the wave equation limit with a first order upwind approximation. We know that the time step restriction is $CFL \leq 1$ when the explicit Euler scheme is applied to this equation Euler time advancement scheme is unconditionally unstable.

3. The results derived in the previous two problems were checked by running the advection-diffusion code (using both central and upwind approximation to the advective term) for $R_\Delta = 0.5, 2.0, 8.0$ on a mesh with 40 equi-spaced points, and adjusting the time step in each case until the solution became unstable. This is somewhat subjective since the error grows for a time in some cases prior to entering an asymptotic decay. In light of this behavior, the additional arbitrary constraint that the rms difference between the computed and exact steady state solution should never exceed the rms difference computed from the initial condition. A bisection search was used to find the find maximum CFL number in each case. The following table summarizes the results. We see that the measured stability bound exceeds the predicted stability bound in all cases tested. The differences are quite small (1.6% and 2.1% for the upwind and central schemes, respectively) for $R_\Delta = 0.5$. The discrepancies grow with $R_\Delta = 0.5$, becoming 8.9% and 50% when $R_\Delta = 8.0$.