# ASEN 5327 Computational Fluid Dynamics
# Spring 2009

## Homework 4 Solution

1. The two-dimensional Euler equations are

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{1}$$

where the vectors $U$, $E$, and $F$ are defined as

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E_t \end{bmatrix} \qquad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E_t + p)u \end{bmatrix} \qquad F = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E_t + p)v \end{bmatrix}$$

The total internal energy is

$$E_t = \rho(e + 1/2(u^2+v^2)) = \rho(c_v T + 1/2(u^2+v^2)) = \frac{p}{\gamma - 1} + 1/2\rho(u^2+v^2)$$

where $\gamma \equiv c_p/c_v$ is the ratio of specific heats. The ideal gas law, expression for the speed of sound, and connection between $R$ $c_p$ and $c_v$ are

$$p = \rho R T, \qquad c^2 = \frac{\gamma p}{\rho}, \qquad R = c_p - c_v.$$

The $x$-direction flux Jacobian is defined as

$$[A] \equiv \frac{\partial E}{\partial U} \tag{2}$$

The easiest way to derive the flux Jacobian is to write the solution vector $U$ symbolically as $\{u_1, u_2, u_3, u_4\}^T$ where $u_1 = \rho$, $u_2 = \rho u$, $u_3 = \rho v$, $u_4 = E_t$. We then write the flux vector in terms of these variables to get

$$E = \begin{bmatrix} u_2 \\ u_2^2/u_1 + (\gamma-1)[u_4 - 0.5(u_2^2 + u_3^3)/u_1] \\ u_2 u_3/u_1 \\ \gamma u_4 u_2/u_1 - 0.5(\gamma-1)(u_2^2+u_3^2)u_2/u_1 \end{bmatrix} \tag{3}$$

It is now a fairly straightforward matter to differentiate with respect to the $u_i$s

$$[A] \equiv \frac{\partial E}{\partial U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ [-u_2^2 + 0.5(\gamma-1)(u_2^2+u_3^2)]/u_1^2 & 2u_2/u_1 - (\gamma-1)u_2/u_1 & -(\gamma-1)u_3/u_1 & (\gamma-1) \\ -u_2u_3/u_1^2 & u_3/u_1 & u_3/u_1 & 0 \\ -\gamma u_4 u_2/u_1^2 + (\gamma-1)(u_2^2+u_3^2)u_2/u_1^3 & \gamma u_4/u_1 - (\gamma-1)u_2^2/u_1^2 - 0.5(\gamma-1)(u_2^2+u_3^2)/u_1^2 & -(\gamma-1)u_2u_3/u_1^2 & \gamma u_2/u_1 \end{bmatrix} \tag{4}$$

This result can be written in terms of the primitive variables through the definition of the $u_i$s. Doing this yields

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \begin{matrix} -0.5(3-\gamma)u^2 + \\ 0.5(\gamma-1)v^2 \end{matrix} & (3-\gamma)u & -(\gamma-1)v & (\gamma-1) \\ -uv & v & u & 0 \\ \begin{matrix} -uc^2/(\gamma-1) - \\ 0.5(2-\gamma)(u^2+v^2)u \end{matrix} & \begin{matrix} c^2/(\gamma-1) + \\ 0.5(3-2\gamma)u^2 + 0.5v^2 \end{matrix} & -(\gamma-1)uv & \gamma u \end{bmatrix} \tag{5}$$

where the definition of the speed of sound, $c^2 = \gamma p/\rho$ has been used.

The eigenvalues of $[A]$ are found by solving $det([A] - \lambda[I]) = 0$. It is easiest to proceed with $[A]$ written symbolically

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ a_{21} & (a_{22}-\lambda) & a_{23} & a_{24} \\ a_{31} & a_{32} & (u-\lambda) & 0 \\ a_{41} & a_{42} & a_{43} & (\gamma u - \lambda) \end{vmatrix} = 0 \tag{6}$$

The determinant is unchanged if any given row in replaced with a linear combination of itself and another row. Some simplification can be achieved in this manner if the third row is replaced with itself minus $u$ times the second row. Doing this results in

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ a_{21} & (a_{22}-\lambda) & a_{23} & a_{24} \\ a_{31} & a_{32} & (u-\lambda) & 0 \\ \tilde{a}_{41} & \tilde{a}_{42} & 0 & (u - \lambda) \end{vmatrix} = 0 \tag{7}$$

where

$$\tilde{a}_{41} = -\left[\frac{c^2}{\gamma-1} - \frac{1}{2}u^2 + \frac{1}{2}v^2\right]u \qquad \tilde{a}_{42} = \left[\frac{c^2}{\gamma-1} - \frac{3}{2}u^2 + \frac{1}{2}v^2\right] - \lambda u \tag{8}$$

The determinant of the modified system is

$$\begin{aligned} &-\lambda[(a_{22}-\lambda)(u-\lambda)(u-\lambda) - a_{23}a_{32}(u-\lambda) - a_{24}\tilde{a}_{42}(u-\lambda)] - \\ &\quad a_{21}(u-\lambda)(u-\lambda) - a_{23}a_{31}(u-\lambda) - a_{24}\tilde{a}_{41}(u-\lambda) = 0 \end{aligned} \tag{9}$$

or upon factoring out the common term $(u - \lambda)$

$$\begin{aligned} &\{-\lambda[(a_{22}-\lambda)(u-\lambda) - a_{23}a_{32} - a_{24}\tilde{a}_{42}] - \\ &\quad a_{21}(u-\lambda) - a_{23}a_{31} - a_{24}\tilde{a}_{41}\}(u-\lambda) = 0 \end{aligned} \tag{10}$$

After substituting the values for the remaining coefficients $a_n$ and simplifying, the characteristic equation becomes

$$(u - c - \lambda)(u + c - \lambda)(u - \lambda)(u - \lambda) = 0 \tag{11}$$

From which the eigenvalues are apparent

$$\lambda = u - c, \quad \lambda = u + c, \quad \lambda = u, \quad \lambda = u \tag{12}$$

2.

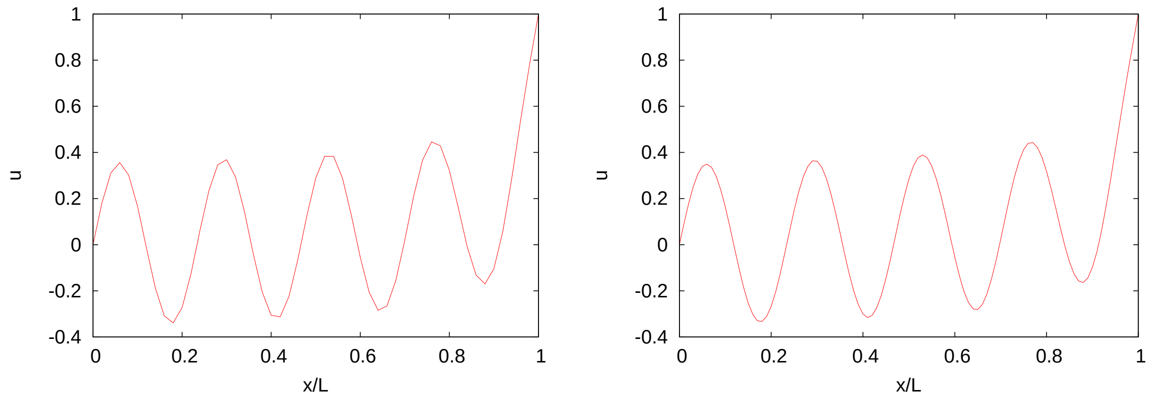


Figure 1: Steady-state solutions for Nx=50 (left) and Nx=100 (right) for $n = 4$.

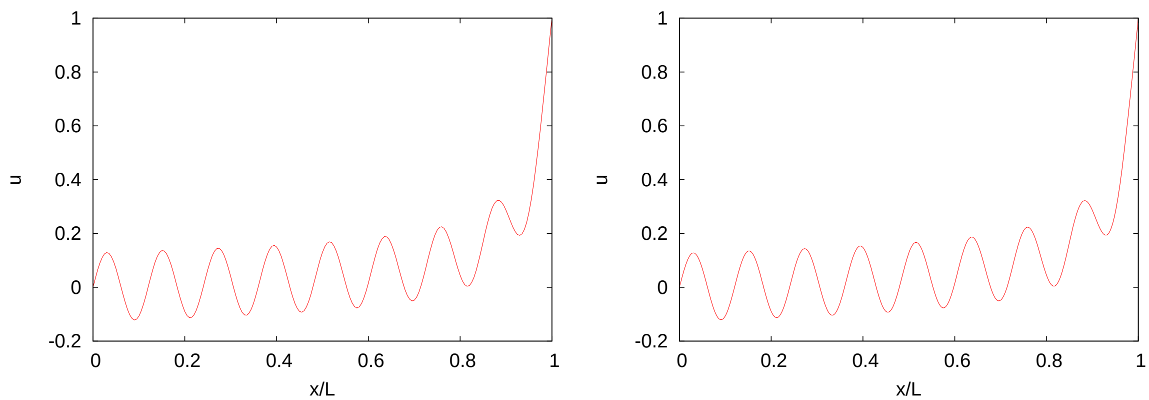


Figure 2: Steady-state solutions for Nx=200 (left) and Nx=500 (right) for $n = 8$.

| Nx,Ny | Number of iterations | CPU Time (sec) | Speed (sec/(Iterations*Nx*Ny)) |
|---|---|---|---|
| 50,50 | 4413 | 0.0580 | $5.256 \times 10^{-7}$ |
| 100,100 | 14829 | 0.745 | $5.024 \times 10^{-7}$ |
| 200,200 | 48071 | 9.435 | $4.907 \times 10^{-7}$ |
| 500,500 | 207461 | 255.154 | $4.920 \times 10^{-7}$ |

Table 1: Computer time required using gfortran on a 2.2GHZ Intel core duo (macbook). Only one core used for the computation.