INTERNAL SOLITARY WAVES

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Abstract

The basic theory of internal solitary waves is reviewed, with the main emphasis on applications to the many observations of such waves in shallow coastal seas, fjords, lakes and in the atmospheric boundary layer. Commencing with the equations of motion for an inviscid, incompressible density-stratified fluid, we describe asymptotic reductions to model long-wave equations, including the well-known Korteweg-de Vries equation, and several extensions. These include a variable-coefficient extended Korteweg-de Vries equation which is proposed as an appropriate evolution equation to describe internal solitary waves in environmental situations. Various analytical and numerical solutions will be discussed.

Introduction

Solitary waves are finite-amplitude waves of permanent form which owe their existence to a balance between nonlinear wave-steepening processes and linear wave dispersion. Typically, they consist of a single isolated wave, whose speed is an increasing function of the amplitude. They are ubiquitous, and in particular internal solitary waves are a commonly occuring feature in the stratified flows of coastal seas, fjords and lakes (see, for instance [1-6]) and in the atmospheric boundary layer (see, for instance [7-9]) Moreover, solitary waves are notable, not only because of their widespread occurrence, but also because they can be described by certain generic nonlinear wave equations which are either integrable, or close to integrability. The most notable example in this context is the now famous Korteweg-de Vries equation, which will figure prominently in this review article.

Our aim here is to describe appropriate model evolution equations to describe internal solitary waves, and indicate briefly some of their more salient properties. In the next section we will demonstrate how canonical model equations can be systematically derived from the complete fluid equations of motion for an inviscid, incompressible, density-stratified, fluid, with boundary conditions appropriate to an oceanic situation. Our main focus is on the Korteweg-de Vries (KdV) equation, but importantly, in order to account for the large amplitudes sometimes observed, we extend this model to the extended Korteweg-de Vries (eKdV) equation which contains both quadratic and cubic nonlinearity, and describe its solitary wave solutions. In the third section, we will introduce the modifications necessary to incorporate the effects of a variable background environment and dissipative processes. The outcome is a variable-coefficient extended Korteweg-de Vries equation. In general this model equation needs to be solved numerically, but to give some insight into the nature of the solutions, we present a particular class of asymptotic solutions describing a slowly-varying solitary wave.

Weakly nonlinear long wave models

Let us consider an inviscid, incompressible fluid which is bounded above by a free surface and below by a flat rigid boundary. Suppose that the flow is two-dimensional and can be described by the spatial coordinates (x, z) where x is horizontal and z is vertical. This configuration is appropriate for the modelling of internal solitary waves in coastal seas, and to some extent in straits, fjords or lakes provided that the effect of lateral boundaries can be ignored. The extensions to this basic model needed to incorporate these lateral effects, the effects of a horizontally variable background state, and various dissipative processes, will be described later.

In the basic state the fluid has density $\rho_0(z)$, a corresponding pressure $p_0(z)$ such that $p_{0z} = -g\rho_0$ describes the basic hydrostatic equilibrium, and a horizontal shear flow in the $x-$direction. Then, in standard notation, the equations of motion relative to this basic state are

$$\rho_0(u_t + u_0 u_x + w u_{0z}) + p_x = -(\rho_0 + \rho)(u u_x + w u_z) - \rho(u_t + u_0 u_x + w u_{0z}), \tag{1a}$$

$$p_z + g\rho = -(\rho_0 + \rho)(w_t + u_0 w_x + u w_x + w w_z) \tag{1b}$$

$$g(\rho_t + u_0 \rho_x) - \rho_0 N^2 w = -g(u\rho_x + w\rho_z) \tag{1c}$$

$$u_x + w_z = 0 \tag{1d}$$

Here $(u_0 + u, w)$ are the velocity components in the (x, z) directions, $\rho_0 + \rho$ is the density, $p_0 + p$ is the pressure and t is time. $N(z)$ is the bouyancy frequency, defined by

$$\rho_0 N^2 = -g\rho_{0z} \tag{2}$$

The boundary conditions are

$$w = 0, \quad \text{at} \quad z = -h \tag{3a}$$

$$p_0 + p = 0, \quad \text{at} \quad z = \eta, \tag{3b}$$

and

$$\eta_t + u_0 \eta_x + u\eta_x = w, \quad \text{at} \quad z = \eta. \tag{3c}$$

Here, the fluid has undisturbed constant depth h, and η is the displacement of the free surface from its undisturbed position $z = 0$.

To describe internal solitary waves we seek solutions whose horizontal length scales are much greater than h, and whose time scales are much greater than N^{-1}. We shall also assume that the waves have small amplitude. Then the dominant balance is obtained by equating to zero the terms on the left-hand side of (1a-d); together with the linearization of the free surface boundary conditions we then obtain the set of equations describing linear long wave theory. To proceed it is useful to use the vertical particle displacement ζ as the primary dependent variable. It is defined by

$$\zeta_t + u_0 \zeta_x + u\zeta_x + w\zeta_z = w. \tag{4}$$

Note that it then follows that the perturbation density field is given by $\rho = \rho_0(z - \zeta) - \rho_0(z) \approx \rho_0 N^2 \zeta$ as $\zeta \to 0$, where we have assumed that as $x \to -\infty$, the density field relaxes to its basic state.

Linear long wave theory is now obtained by omitting the right-hand side of equations (1a-d), and simultaneously linearising boundary conditions (3b,c). Solutions are sought in the form

$$\zeta = A(x - ct)\phi(z), \tag{5}$$

while the remaining dependent variables are then given by analogous expressions. Here c is the linear long wave speed, and the modal function $\phi(z)$ is defined by the boundary-value problem,

$$\{\rho_0(c - u_0)^2 \phi_z\}_z + \rho_0 N^2 \phi = 0, \quad \text{in} \quad -h < z < 0, \tag{6a}$$

$$\phi = 0 \quad \text{at} \quad z = -h, \tag{6b}$$

$$\text{and} \qquad (c - u_0)^2 \phi_z = g\phi \quad \text{at} \quad z = 0. \tag{6c}$$

Typically, the boundary-value problem (6a-c) defines an infinite sequence of modes, $\phi_n^{\pm}(z)$, $n = 0, 1, 2, \ldots$, with corresponding speeds $c_n^{\pm}$. Here, the superscript "$\pm$" indicates waves with $c_n^+ > \max u_0(z)$ and $c_n^- < \min u_0(z)$ respectively. We shall confine our attention to these regular modes, and consider only stable shear flows. Note that it is useful to let $n = 0$ denote the surface gravity waves for which c scales with $\sqrt{gh}$, and then $n = 1, 2, 3, \ldots$ denotes the interval gravity waves for which c scales with Nh. In general, the boundary-value problem (6a-c) is readily solved numerically. Typically, $\phi_n^{\pm}(z)$, $n = 1, 2, 3, \ldots$, have n extremal points in the interior of the fluid, and vanish near $z = 0$ (and, of course, also at $z = -h$)

It can now be shown that, within the context of linear long wave theory, any localised initial disturbance will evolve into a set of outwardly propogating modes, each given by an expression of the form (5). Assuming thats the speeds $c_n^{\pm}$ of each mode are sufficiently distinct, it is sufficient for large times to consider just a single mode. Henceforth, we shall omit the indices and assume that the relevant mode (usually n=1) has speed c, amplitude A and modal function $\phi(z)$. Then, as time increases, we expect the hitherto neglected nonlinear terms to have an effect, and to cause wave steepening. However, this is opposed by the terms representing linear wave dispersion, also neglected in the linear long wave theory. A balance between these two effects emerges as time increases and the well-known outcome is the Korteweg-de Vries (KdV) equation, or a related equation, for the wave amplitude.

The formal derivation of the evolution equation requires the introduction of the small parameters, α and ϵ, respectively characterising the wave amplitude and dispersion. A KdV balance requires $\alpha = \epsilon^2$, with a corresponding timescale of ϵ^{-3}. The asymptotic analysis required is well understood (see e.g. [3-4,10-13], so we shall give only a brief outline here. We introduce the scaled variables

$$\tau = \epsilon\alpha t \ , \quad \theta = \epsilon(x - ct) \tag{7}$$

and then let

$$\zeta = \alpha A(\theta, \tau)\phi(z) + \alpha^2 \zeta_2 + \ldots , \tag{8}$$

with similar expressions analgous to (8) for the other dependent variables. At leading order, we get the linear long wave theory for the modal function $\phi(z)$ and the speed c, defined by (6a-c). Note that since the modal equation is homogeneous, we are free to impose a normalization condition on $\phi(z)$. A commonly used condition is that $\phi(z_m) = 1$ where $|\phi(z)|$ achieves a maximum value at $z = z_m$. In this case the amplitude αA is uniquely defined as the amplitude of ζ (to $0(\alpha)$) at the depth z_m. However, other normalisations are possible and sometimes to be preferred. Then, at the next order, we obtain the equation for ζ_2,

$$\{\rho_0(c - u_0)^2 \zeta_{2\theta z}\}_z + \rho_0 N^2 \zeta_{2\theta} = M_2, \quad \text{in} \quad -h < z < 0, \tag{9a}$$

$$\zeta_{2\theta} = 0, \quad \text{at} \quad z = -h, \tag{9b}$$

$$\rho_0(c - u_0)^2 \zeta_{2\theta z} - \rho_0 g \zeta_{2\theta} = N_2, \quad \text{at} \quad z = 0. \tag{9c}$$

Here the inhomogeneous terms M_2, N_2 are known in terms of $A(\theta, \tau)$ and $\phi(z)$, and are given by

$$M_2 = 2\{\rho_0(c - u_0)\phi_z\}_z A_\tau + 3\{\rho_0(c - u_0)^2 \phi_z^2\}_z A A_\theta - \rho_0(c - u_0)^2 \phi A_{\theta\theta\theta}, \tag{10a}$$

$$N_2 = 2\{\rho_0(c - u_0)\phi_z\} A_\tau + 3\{\rho_0(c - u_0)^2 \phi_z^2\} A A_\theta. \tag{10b}$$

Note that the left-hand side of the equations (9a-c) is identical to the equations defining the modal function (i.e. (6a-c)), and hence can be solved only if a certain compatibility condition is satisfied, given by,

$$\int_{-h}^{0} M_2 \phi \, dz = [N_2 \phi]_{z=0} \tag{11}$$

Note that the solution for ζ_2 contains a term $A_2\phi(z)$ where the amplitude A_2 is left undetermined at this stage.

Substituting the expressions (10a,b) into (11) we obtain the required evolution equation for A, namely the KdV equation

$$A_\tau + \mu A A_\theta + \lambda A_{\theta\theta\theta} = 0. \tag{12}$$

Here, the coefficients μ and λ are given by

$$I\mu = 3\int_{-h}^{0} \rho_0 (c-u_0)^2 \phi_z^3 \, dz, \tag{13a}$$

$$I\lambda = \int_{-h}^{0} \rho_0 (c-u_0)^2 \phi^2 \, dz, \tag{13b}$$

where

$$I = 2\int_{-h}^{0} \rho_0 (c-u_0) \phi_z^2 \, dz. \tag{13c}$$

Confining attention to waves propagating to the right, so that $c > u_M = \max u_0(z)$, we see that I and λ are always positive. Further, if we normalise the first internal modal function $\phi(z)$ so that it is positive at its extremal point, then it is readily shown that for the usual situation of a near-surface pycnocline, μ is negative for this first internal mode. However, in general μ can take either sign, and in some special situations may even be zero. Explicit evaluation of the coefficients μ and λ requires knowledge of the modal function, and hence they are usually evaluated numerically.

Proceeding to the next highest order will yield an equation set analogous to (9a-c) for ζ_3, whose compatibility condition then determines an evolution equation for the second-order amplitude A_2. We shall not give details here (see [14]), but note that using the transformation $A + \alpha A_2 \to A$, and then combining the KdV equation (12) with the evolution equation for A_2 will lead to a higher-order KdV equation for A of the form.

$$A_\tau + \mu A A_\theta + \lambda A_{\theta\theta\theta} + \alpha(\nu A^2 A_\theta + \beta A_{\theta\theta\theta\theta\theta} + \gamma A A_{\theta\theta\theta} + \delta A_\theta A_{\theta\theta}) = 0. \tag{14}$$

However, we must now point out that this higher-order Korteweg-de Vries equation (14) is, strictly speaking, an asymptotic result valid when α is sufficiently small, and is most likely to be useful when the coefficient μ of the quadratic nonlinear term is small (e.g. $O(\epsilon)$ where we recall that $\alpha = \epsilon^2$). Nevertheless, because observed internal solitary waves are often quite large, it may be useful to use (14) as the model equation even when μ is not small.

Next we note that choosing a particular normalisation fixes the amplitude A_2 and hence ζ_2 uniquely, and then the coefficients in the higher-order Korteweg-de Vries equation (14) are likewise uniquely determined. But a different choice for this normalisation would produce a different set of coefficients. This allows us to make a near-identity transformation

$$B = A + \alpha\{\frac{1}{2} a A^2 + b A_{\theta\theta}\}, \tag{15}$$

where a specific choice of the coefficients a, b corresponds to a change in the normalisation. It can now be readily verified that substitution of (15) into (14) allows us to generate asymptotically a higher-order Korteweg-de Vries equation for B of the same form as (14), but with the coefficients ν, β, γ and δ replaced by $\nu - a\mu/2, \beta, \gamma$ and $\delta - 3a\lambda + 2b\mu$ respectively. Note that the coefficients μ, λ in the first-order Korteweg-de Vries equation are not changed, and nor are the coefficients β and γ of the higher-order equation.

In particular, we can now choose a so that the higher-order Korteweg-de Vries equation for B is Hamiltonian. For (14) to be exactly Hamiltonian (as opposed to being only asymptotically

Hamiltonian), it is necessary that $\delta = 2\gamma$ which is generally not the case (see [14]). However the near-identity transformation (15) with the choice $3a\lambda = \delta - 2\gamma$ and $b = 0$ will produce a Hamiltonian form. With $\delta = 2\gamma$ the Hamiltonian form for (14) is,

$$A_\tau = -\frac{\partial}{\partial\theta}\frac{\delta H}{\delta A}, \tag{16}$$

where the Hamiltonian H has the density

$$\frac{1}{6}\mu A^3 - \frac{1}{2}\lambda A_\theta^2 + \alpha\left(\frac{1}{12}\nu A^4 + \frac{1}{2}\beta A_{\theta\theta}^2 - \frac{1}{2}\gamma A A_\theta^2\right). \tag{17}$$

When (14) is Hamiltonian then it conserves not only the mass (i.e the integral of A), but also the momentum whose density is A^2, as well as the Hamiltonian itself, whose density is H. For numerical purposes it is desirable that the evolution equation (14) should be Hamiltonian, and hence the renormalisation implied by (15) is generally recommended.

Further, more is possible when the near-identity transformation (14) is enhanced to

$$B = A + \alpha\{\frac{1}{2}aA^2 + bA_{\theta\theta} \ + a^{'}A_\theta\int^\theta A\,d\theta + b^{'}\theta A_\tau\}. \tag{18}$$

It can then be shown that, provided only that μ, λ are not zero, it is possible to choose the available constants $a, b, a^{'}, b^{'}$ so that the resulting equation for B is just the Korteweg-de Vries equation (i.e has the form (12)), or alternatively, the coefficients $\nu, \beta, \gamma, \delta$ in (15) are all zero) with an error of $O(\alpha^3)$ (see [15-17]) Thus, in general the higher-order equation (14) is asymptotically reducible to the integrable Korteweg-de Vries equation (12). However, we hasten to point out that although this is an intriguing result, its use in practice in this context may well be very limited because the amplitude parameter α is not necessarily so small that the transformation (15) is applicable, and also there are circumstances when the coefficient μ is close to zero, in which case this reduction is not possible.

Indeed, a particularly important special case of the higher-order KdV equation (14) arises when precisely when the nonlinear coefficient μ (13a) in the KdV equation is close to zero. In this situation, the cubic nonlinear term in the higher-order KdV equation is the most important higher-order term, and (14) reduces to the extended KdV equation,

$$A_\tau + \mu AA_\theta + \alpha\nu A^2A_\theta + \lambda A_{\theta\theta\theta} = 0. \tag{19}$$

For $\mu \approx 0$, a rescaling is needed and the optimal choice is to assume that μ is $0(\epsilon)$, and then replace A with A/ϵ. In effect the amplitude parameter is ϵ in place of ϵ^2.

In some atmospheric and oceanic applications, the depth h is not necessarily small relative to the horizontal length scale of the solitary wave, but nevertheless the density stratification is effectively confined to a thin layer of depth h_1, which is much shorter that the horizontal length scales. In this case, a different theory is needed, and was first developed in [18-19] The outcome is the intermediate long-wave (ILW) equation ([8,20]), and for further details we refer the reader to [3-4].

Both of the evolution equations, viz. the KdV equation (12) and the extended KdV equation (19) are exactly integrable, (see, for instance, [21]), with the consequence that the initial-value problem with a localised initial condition is exactly solvable. The most important implication of this integrability for the KdV equation is that an arbitrary initial disturbance will evolve into a finite number (M) of solitary waves (called solitons in this context) and an oscillatory decaying tail. This, together with the robust stability properties of solitary waves, explains why internal solitary waves are so commonly observed. Note that because solitary waves typically have speeds which increase with the wave amplitude, the M waves are rank-ordered by amplitude as $t \to \infty$.

Also, to produce solitary waves at all, the initial disturbance should have the correct polarity (e.g. $\mu \int_{-\infty}^{\infty} A(\theta, 0) d\theta > 0$ for the case of the KdV equation (12)). Note that, in applications the initial condition $A(\theta, 0)$ for the evolution equation is found by first solving the linear long wave equations, and then identifying the mode of interest. Thus $A(\theta, 0)$ is given by (5), which in turn is a reduction from the actual initial conditions.

It follows from the proceeding discussion that in describing the solution of the evolution equations, the most important step is to determine the solitary wave solutions. For the KdV equation (12) this is given by

$$A = a \operatorname{sech}^2 \beta(\theta - V\tau), \tag{20a}$$

$$\text{where} \qquad V = \frac{1}{3}\mu a = 4\lambda\beta^2. \tag{20b}$$

Note that the speed V is for the phase variable θ, and the actual total speed is $c + \alpha V$. Since the dispersion coefficient λ is always positive for right-going waves, it follows that these solitary waves are always supercritical ($V > 0$), and are waves of elevation or depression according as $\mu \gtrless 0$. We also see that β^{-1} is proportional to $|a|^{-\frac{1}{2}}$, and hence the larger waves are not only faster, but narrower.

For the extended KdV equation (19) the corresponding solitary wave is given by [22-23],

$$\epsilon A = \frac{a}{b + (1-b)\cosh^2 \beta(\theta - V\tau)}, \tag{21a}$$

$$\text{where} \qquad V = \frac{1}{3}a\left(\frac{\mu}{\epsilon} + \frac{1}{2}\nu a\right) = 4\lambda\beta^2, \tag{21b}$$

and

$$\text{and} \qquad b = \frac{-\nu a}{\left(\frac{2\mu}{\epsilon} + \nu a\right)}. \tag{21c}$$

Here we recall that μ is $0(\epsilon)$ in (19), and so A is rescaled to ϵA. There are two cases to consider. If $\nu < 0$, then there is a single family of solutions such that $0 < b < 1$ and $\mu a > 0$. As b increases from 0 to 1, the amplitude $|a|$ increases from 0 to a maximum of $|\mu/\epsilon\nu|$, while the speed V also increases from 0 to a maximum of $-\mu^2/6\epsilon^2\nu$. In the limiting case when $b \to 1$ the solution (21a) describes the so-called "thick" solitary wave, which has a flat crest of amplitude $a_m = -\mu/\epsilon\gamma$ and is terminated at each end by the bore-like solutions

$$\epsilon A = \frac{1}{2}a_m\{1 \mp \tanh \beta_m(\theta - V_m\tau)\} \tag{22}$$

where $V_m = -\mu^2/6\epsilon^2\nu = 4\lambda\beta_m^2$. In the case $\mu/\epsilon = 0$ there are no exact solitary wave solutions of (19) when $\nu < 0$, but instead there is the travelling bore solution

$$\epsilon A = a \tanh \beta(\theta - V\tau), \tag{23a}$$

$$\text{where} \qquad V = \frac{1}{3}\nu a^2 = -2\lambda\beta^2. \tag{23b}$$

Note here that the amplitude of the travelling bore is a free parameter, and that the speed $V < 0$.

For the case when $\nu > 0$, $b < 0$ and there are two families of solitary waves. One is defined by $-1 < b < 0$, has $\mu a > 0$, and as b decreases from 0 to -1, the amplitude a increases from 0 to ∞, while the speed V also increases from 0 to ∞. The other is defined by $-\infty < b < -1$, has $\mu a < 0$ and, as b increases from $-\infty$ to -1, the amplitude $|a|$ increases from $-2|\mu|/\epsilon\nu$ to ∞. In this case solitary waves exist if $\mu/\epsilon = 0$ ($b = -1$) and are given by

$$\epsilon A = a \operatorname{sech} 2\beta(\theta - V\tau), \tag{24a}$$

$$\text{where} \qquad V = \frac{1}{6}\nu a^2 = 4\lambda\beta^2 \tag{24b}$$

On the other hand, as $b \to -\infty$, $\beta \to 0$ and the solitary wave (21a) reduces to the algebraic form

$$\epsilon A = \frac{a_0}{1 + \nu a_0^2 \theta^2 / 24\lambda}, \quad a_0 = -\frac{2\mu}{\epsilon\nu}. \tag{25}$$

For the higher-order KdV equation (14) the structure of the solitary wave solutions has not yet been determined completely, and indeed if the coefficients λ, β of the linear dispersive terms happen to have the same sign, then there is the possibility that there may not exist any "pure" solitary waves, and instead localised structures are accomanpanied by ripples at infinity. Such solutions are called *generalised* solitary waves, and for a general acount of their properties, the reader is referred to [24].

Variable background

The KdV equation (12) is the basic model for the situation studied in Section 2, when the flow is unidirectional, and the background state is horizontally uniform. Our purpose now is to extend this basic model to situations where there is a variable background environment. This can arise due to a variable depth $h(X)$, or due to horizontal variability in the basic density $\rho_0(X; z)$ and horizontal velocity field $u_0(X; z)$ where $X = \epsilon\alpha x$. Here, for simplicity, we are considering the situation when the background variability is unidirectional and in the flow direction. The scaling indicates that we are assuming that the background varies on a length scale which is much greater than that of the solitary waves, but is comparable to the length scale over which the wave field evolves. The modal functions $\phi(X; z)$ are again defined by (6a-c), but now depend parametrically on X, and hence so does the wave speed $c(X)$. An asymptotic expansion analogous to (8) is then introduced, but the variables τ and θ in (7) are here replaced by

$$s = \int_0^X \frac{dX'}{c(X')}, \qquad \psi = \frac{1}{\alpha}(s - \tau), \tag{26}$$

where we recall that τ is defined by (7). The amplitude $A(s, \psi)$ can then be shown to satisfy the variable-coefficient extended KdV equation [25-26],

$$A_s + \frac{\sigma_s}{2\sigma}A + \frac{\mu}{c}AA_\psi + \frac{\nu}{c}A^2 A_\psi + \frac{\lambda}{c^3}A_{\psi\psi\psi} + \gamma\mathcal{D}(A) = 0, \tag{27}$$

which thus replaces (19). Here the coefficients $\mu(X)$, $\lambda(X)$ are defined by (30a,b), $\sigma(X) = c^2 I$ and the term $\alpha\mathcal{D}(A)$ represents the effects of friction. The significance of the coefficient σ is that σA^2 is a measure of the wave action flux in the $X-$direction, and is a conserved quantity in the absence of dissipation.

Dissipation can arise from several sources, such as turbulent mixing in the fluid interior associated with local shear unstability, scattering due to bottom roughness and viscous decay due to the bottom boundary layer. Each of these can be modelled by letting

$$\mathcal{D}(A) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-ik)^m \exp(ik\psi)\mathcal{F}(A)dk, \tag{28}$$

where $\mathcal{F}(A)$ is the Fourier transform of A. The index "m" determines the type of dissipation; $m = 2$ leads to the KdV-Burgers equation since then $\mathcal{D}(A) = A_{\psi\psi}$ (with $\gamma < 0$), while $m = 0$ (with $\alpha > 0$) corresponds to linear Rayleigh damping. For a laminar bottom boundary layer $m = \frac{1}{2}$ and for a turbulent bottom boundary layer, it is customary to replace the expression (3.3) with

$$\mathcal{D}(A) = |A|A, \tag{29}$$

Next we incorporate the effects of wave diffraction, for the case when these are relatively weak. That is, relative to the dominant $X-$ direction there is a weak tendency for the waves to spread in the transverse $y-$direction. The appropriate evolution equation is then

$$\{A_s + \frac{\sigma_s}{2\sigma}A + \frac{\mu}{c}AA_\theta + \frac{\nu}{c}A^2A_\psi + \frac{\lambda}{c^3}A_{\psi\psi\psi} + \gamma\mathcal{D}(A)\}_\psi + \frac{1}{2}c^2(A_{YY} - \Gamma^2 A) = 0, \qquad (30a)$$

$$\text{where} \qquad Y = \epsilon^2 y, \text{ and } \Gamma = \frac{f}{\epsilon^2 c} \qquad (30b)$$

Here we have also included the effects of the earth's rotation (see [25]), represented by the local value of the Coriolis parameter f, since the internal Rossby radius $|c/f|$ may well be comparable with other transverse scales. Note that fh/c is required to be at least $0(\epsilon^2)$. In the absence of these rotation terms and the dissipative term, equation (30a) is a variable-coefficient Kadomtsev-Petviashvili (KP) equation. Further, in the absence of any background variation so that (30a) has constant coefficients, the KP equation is an integrable equation when $\nu = 0$, and is generally accepted as an appropriate two-dimensional generalisation of the KdV equation.

More generally, when the background environment varies in both spatial directions, and through the basic velocity and density fields, possibly in time also, an evolution equation analogous to (30a) can be derived [26]. In this very general situation, the modal functions and the speed c depend parametrically on slow time and horizontal spatial variables. The speed c is then used to determine space-time rays which in turn then determine the dominant direction for the wave propagation, so that s is a time-like variable along this ray, ψ is a phase variable describing the wave structure and Y is a co-ordinate transverse to the ray. The counterpart of the generalised KdV equation (27), or its two-dimensional counterpart, the generalised KP equation (30a) can also be derived for deep fluids [27], thus providing the appropriate extension of the ILW equation.

In general the gKdV equation (27) (or its two-dimensional counterpart (30a)) must be solved numerically. However, to gain insight into the expected behaviour of the solitary wave solutions, it is useful to consider the asymptotic construction of the slowly-varying solitary wave solution, in which it is assumed that the background variability and the dissipative effects are sufficiently weak that a solitary wave is able to maintain its structure over long distances. For simplicity, we consider here only the case when the cubic nonlinear term in (27) can be ignored, and so put $\nu = 0$ hereafter. In this case a multi-scale perturbation technique [28-29] can be used in which the leading term is

$$A \approx a\text{sech}^2\beta(\psi - \int_0^s V ds) \qquad (31a)$$

$$\text{where} \qquad V = \frac{\mu a}{3c} = \frac{4\lambda\beta^2}{c^3}. \qquad (31b)$$

Here the wave amplitude $a(s)$, and hence also $V(s)$, $\beta(s)$, are slowly-varying functions of s. Their variation is most readily determined by noting that (27) possesses an "energy" law,

$$\frac{\partial}{\partial s}\int_{-\infty}^{\infty}\sigma A^2 d\psi + 2\sigma\gamma\int_{-\infty}^{\infty}A\mathcal{D}(A)d\psi = 0, \qquad (32)$$

which expresses conservation of wave action flux in the absence of dissipation. Substitution of (31a) into (32) gives

$$\frac{\partial}{\partial s}\left(\frac{2}{3}\frac{\sigma a^2}{\beta}\right) + \sigma\gamma a\int_{-\infty}^{\infty}\text{sech}^2\beta\psi\mathcal{D}(a\text{sech}^2\beta\psi)d\psi = 0. \qquad (33)$$

Using the relations (31b) this is an equation for $a(s)$. However, although the slowly-varying solitary wave conserves "energy" it cannot simultaneously conserve mass. Instead, it is accompanied by a trailing shelf of small amplitude but long length scale whose amplitude A_- at the rear of the solitary wave is given by

$$\frac{\mu A_-}{3c} = \frac{\beta_s}{\beta^2} + \frac{\gamma}{a}\int_{-\infty}^{\infty}\left\{\frac{3}{2}\text{sech}^2\beta\psi - 1\right\}\mathcal{D}(a\text{sech}^2\beta\psi)d\psi. \tag{34}$$

When the coefficients $\lambda, \mu, \gamma, \sigma$ and c are known explicitly as functions of s, the expressions in (33) and (34) can also be readily evaluated explicitly. However, usually these coefficients, being determined *inter alia* from the modal functions, are known only numerically, and hence $a(s)$ and $A_-(s)$ can also only be obtained numerically.

In the absence of any dissipation (i.e. $\gamma = 0$) equation (33) shows that $\sigma a^2\beta^{-1}$ is a constant on the ray path, and hence, on using the relations (31b) we see that

$$a^3 \propto \mu c^2/\lambda\sigma^2 \tag{35}$$

gives an explicit formula for $a(s)$. Further, in this case (3.10) shows that if the wave width β^{-1} increases (decreases) along the ray path, then the trailing shelf amplitude A_- has the opposite (same) polarity to the solitary wave. A situation of particular interest occurs when the coefficient $\mu(s)$ changes sign at some particular location, say $s = s_0$. In the oceanic environment this commonly occurs as the depth h of the ocean decreases, where μ is typically negative in the deeper water (here we consider waves propagating to the right so that $I(13c) > 0$). In this case, since the dispersive coefficient λ is always positive (13b), it follows from (31b) that the solitary wave is a wave of depression when $\mu < 0$, but a wave of elevation when $\mu > 0$. The issue then arises as to how the solitary will behave as $\mu \to 0$ (i.e. as $s \to s_0$), and in particular, as to whether a solitary wave of depression can be converted into one or more solitary waves of elevation as the critical point $s = s_0$ is traversed. This problem has been intensively studied (see, for instance [30] and the references therein), and the solution depends on how rapidly the coefficient μ changes sign. If μ passes through zero rapidly compared to the local width of the solitary wave, then the solitary wave is destroyed, and converted into an oscillatory wavetrain. On the other hand, if μ changes sufficient slowly that the formula (35) holds, we see that as $\mu \to 0$ so does $a \to 0$ in proportion to $|\mu|^{\frac{1}{3}}$, while $\beta \to 0$ as $|\mu|^{\frac{2}{3}}$, and $A_- \to \infty$ as $|\mu|^{-\frac{8}{3}}$. Thus, as the solitary wave amplitude decreases, the amplitude of the trailing shelf, which has the opposite polarity, grows indefinitely until a point is reached just prior to the critical point where the slowly-varying solitary wave asymptotic theory fails. A combination of this trailing shelf and the distortion of the solitary wave itself then provide the appropriate "initial" condition for one or more solitary waves of the opposite polarity to emerge as the critical point $s = s_0$ is traversed. However, these conclusions depend on the cubic nonlinear term in (27) being negligible in the vicinity of $s = s_0$. When this is not the case the outcome depends on the sign of ν at $s = s_0$. If $\nu > 0$ so that solitary waves of either polarity can exist when $\mu = 0$, then the solitary wave preserves its polarity (i.e. remains a wave of depression) as the critical point is traversed. On the other hand if $\nu < 0$ so that no solitary wave can exist when $\nu = 0$ then the solitary wave of depression may be converted into one or more solitary waves of elevation, or into a lreather solution, or into an oscillatory wavetrain; for more details of this case, see [30].

Next we use equation (33) to determine the effects of dissipation. Here we assume that the background is uniform so that the coefficients $\sigma, \mu, \lambda, \gamma$ and c are all constants. Then, for the case of a laminar bottom boundary layer $m = \frac{1}{2}$ in (28) and equation (33) can be solved for a to give

$$a = a_0\left\{1 + 0.168\left(\frac{4\beta_0^2}{3}\right)^{\frac{1}{4}}\gamma s\right\}^{-4} \tag{36}$$

where $a_0 = a(0)$ is the initial value of the amplitude a and β_0 is related to a_0 through the expression (31b). On the other hand, for a turbulent bottom boundary layer the expression (29) should be used in (33) which leads to the expression

$$a = a_0 \left\{ 1 + \frac{16}{15} a_0 \gamma s \right\}^{-1} . \tag{37}$$

For typical oceanic parameters both these expressions give life times which are several orders of magnitude greater than the wave's intrinsic time scale.

The effects of the earth's rotation is described by equation (30a) where Γ represents the Coriolis parameter; more precisely $(\epsilon^2\Gamma)^{-1}$ is the internal Rossby radius. In the absence of any transverse dependence (i.e. $A = A(s, \psi)$ and $A_Y \equiv 0$) equation (30a) is often called the Ostrovsky equation [12] and has been intensively studied. Even for the case of a uniform background (i.e. the coefficients σ, μ, λ and c are all constants) and no dissipation (i.e. $\gamma = 0$), the Ostrovsky equation possesses no solitary wave solutions when $\Gamma \neq 0$ [32]. Instead a localised initial condition decays with the radiation of Poincaré waves. For sufficiently small values of Γ, the decay of a solitary wave due to this radiation can be calculated explicitly using the slowly-varying solitary wave formulation [32], and it is found that

$$a = a_0(1 - c^2\Gamma^2 s/2\beta_0)^2 \tag{38}$$

where we recall that a_0 is the initial amplitude and β_0 is related to a_0 by (31b). The formula (38) predicts the extinction of the solitary wave in finite time. For typical oceanic parameters, this extinction time is comparable with the life times due to dissipation, and hence the Coriolis effect is a candidate for the eventual decay of oceanic internal waves.

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