2D vortex method

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1 Inviscid vortex approximations

Vortex methods are based on an inviscid flow assumption. In two dimensions the inviscid vorticity equation indicates that vortex elements are simply advected with the flow. This fact forms the basis of vortex methods where individual vortex elements are advanced in time using the kinematic relations

$$\frac{dx_j}{dt} = u_j, \tag{1}$$

$$\frac{dy_j}{dt} = v_j, \tag{2}$$

where (x_j, y_j) are the coordinates of the j^{th} vortex and (u_j, v_j) is the velocity at this location. Each vortex element induces a velocity field around it and thus (u_j, v_j) are found by summing the induced velocities from all vortices in the system. Thus the first task is to develop a general expression for the velocity field induced by a single vortex. We will start by considering an idealized vortex which has all of its vorticity concentrated in a singular fashion at its center. Later we will improve the model by distributing the vorticity over a small region near the center, as must be the case for a real vortex.

For two-dimensional idealized flow, it is most convenient to perform analysis using complex variables. We start by defining a complex valued potential function

$$F(z) = \phi(x, y) + i\psi(x, y), \tag{3}$$

where $z = z + iy$, and where $\phi(x, y)$ and $\psi(x, y)$ are the usual velocity potential and streamfunction, each of which satisfies Laplace's equation. The velocity components are related to derivatives of the potential and streamfunction via

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \tag{4}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{5}$$

These relations, in conjunction with the definition of the complex potential given above indicate that

$$W(z) = \frac{dF}{dz} = \frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv. \tag{6}$$

Here we have used the invariance of the derivative direction in the complex plane to replace the z derivative with an x derivative. The same end result can be derived by taking the derivative direction to be iy. The above result can be transformed into polar coordinates to give

$$W(z) = (u_r - iu_\theta)e^{-i\theta}. \tag{7}$$

The complex potential for a point vortex located at position z_j and with counter clockwise (positive) azimuthal velocity is

$$F(z) = \frac{-i\Gamma}{2\pi} \ln(z - z_j), \tag{8}$$

which has corresponding velocity

$$W(z) = \frac{dF}{dz} = \frac{-i\Gamma}{2\pi(z - z_j)} = \frac{-i\Gamma}{2\pi r_{z_j}} e^{-i\theta_{z_j}}, \tag{9}$$

where r_{z_j} and θ_{z_j} are the the radius and azimuthal angle measured relative to the point z_j. Comparing with Eq. (7), we see that the vortex has only an azimuthal velocity component about the point z_j, which is given by

$$u_\theta(r_{z_j}) = \frac{\Gamma}{2\pi r_{z_j}}. \tag{10}$$

or, equivalently in Cartesian coordinates

$$(u_j, v_j) = \frac{\Gamma}{2\pi} \left(\frac{-(y - y_j), \quad (x - x_j)}{(x - x_j)^2 + (y - y_j)^2} \right) \tag{11}$$

In 2D the vorticity vector has a single component, which is directed normal to the plane of motion. In polar coordinates, this component is

$$\omega = \frac{1}{r} \left(\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right), \tag{12}$$

and when applied to the velocity of a vortex at the origin (Eq. (10) with $z_j = 0$), we have

$$\omega = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\Gamma}{2\pi r} \right). \tag{13}$$

For $r \neq 0$ the above expression evaluates to zero. The behavior when $r = 0$ is difficult to asses, however, due to the indeterminate form. The easiest way to investigate the behavior in this limit is to make use of Stokes theorem which, when applied to the velocity vector, states

$$\int_0 \nabla \times \vec{u} \, dA = \oint \vec{u} \cdot \vec{dl} \tag{14}$$

$$\int_0 \omega \, dA = \int_0^{2\pi} \left(\frac{\Gamma}{2\pi r} \right) r d\theta \tag{15}$$

$$= \Gamma. \tag{16}$$

This result shows that the circulation around the vortex is independent of r and is equal to Γ. Furthermore, since $\omega = 0$ when $r \neq 0$, all of the contribution to the integral of vorticity must come from the center. Mathematically this means that the vorticity is a delta function, being zero everywhere except at the vortex center, where its value is infinite. This fact is important in the context of the irrotational assumption in the idealized flow approximation used here. It turns out that the approximation is still valid if the vorticity differs from zero only at a collection of fixed points having no net area. Basically Laplace's equation is accepting of such singular points.

2 Periodic boundary conditions

While Eq. (11) can be used as the basis of a vortex method applied in an unbounded space, it is often the case that periodic boundary conditions are desired instead. As an example consider a simple two-dimensional vortex sheet. If the sheet extends to $\pm\infty$ in the flow direction then an infinite number of discrete vortices would be needed to describe it. The sheet could be truncated at far distances from the center, but edge effects would then play some role in the solution. The usual remedy for this problem is to consider instead a finite segment of sheet of width x_L that is then replicated an infinite number of times in a periodic fashion. Although there are still an infinite number of vortices in the discrete representation, nearly all of these are simply related to a manageable number distributed along the sheet over the interval $[-x_L/2, x_L/2]$. Making use of Eq.

(9), the net velocity due to the vortex of strength Γ_j at position $z_j = x_j + iy_j$ along with its infinite collection of periodic images lying at $z_j + nx_L = x_j + nx_L + iy_j$, $n = \pm 1 \ \pm 2 \ \pm 3 \ldots \pm \infty$ is

$$W_j(z) = -\frac{i\Gamma_j}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{z - (z_j + nx_L)} \tag{17}$$

The sum above can be identified with the series expansion for the cot function[?], which results in the surprisingly simple form

$$W_j(z) = -\frac{i\Gamma_j}{2x_L} \cot(\pi(z - z_j)/x_L) \tag{18}$$

Another way to arrive at this result is to use conformal mapping. If two spaces are related via $\zeta = f(z)$ then the velocities are related via

$$W(z) = \frac{dF}{dz} = \frac{dF}{d\zeta}\frac{d\zeta}{dz} = W(\zeta)\frac{d\zeta}{dz} \tag{19}$$

Note that the transformation

$$\zeta = \frac{x_L}{\pi} \sin(\pi(z - z_j)/x_L) \tag{20}$$

maps the point $\zeta = 0$ to the points $z = z_j + nx_L$, $n = 0 \ \pm 1 \ \pm 2 \ldots$. Thus a vortex at the origin in the ζ plane maps to a horizontal periodic array of vortices along the line $y = y_j$ in the z plane. Using Eq. (19), the velocity due to the periodic array is

$$\begin{aligned} W_j(z) = W(\zeta)\frac{d\zeta}{dz} = -\frac{i\Gamma_j}{2\pi\zeta}\frac{d\zeta}{dz} \quad &= \quad -\frac{i\Gamma_j}{2\pi}\frac{\pi}{x_L \sin(\pi(z - z_j)/x_L)} \cos(\pi(z - z_j)/x_L) \\ &= \quad -\frac{i\Gamma_j}{2x_L} \cot(\pi(z - z_j)/x_L) \end{aligned} \tag{21}$$

By writing $z = x + iy$, $\cot(z) = \cos(z)/\sin(z)$, $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$, $\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$, and then performing slightly tedious algebra, it is possible to write the above expression in Cartesian component form as

$$(u_j, v_j) = \frac{\Gamma_j}{2x_L} \left(\frac{-\sinh\left[\frac{2\pi(y-y_j)}{x_L}\right], \quad \sin\left[\frac{2\pi(x-x_j)}{x_L}\right]}{\cosh\left[\frac{2\pi(y-y_j)}{x_L}\right] - \cos\left[\frac{2\pi(x-x_j)}{x_L}\right]} \right) \tag{22}$$

A simple vortex model can be created by combining Eqs. (2) with either Eq. (11) (for non-periodic boundaries) or (22) (for periodic boundary conditions in x). We start by defining positions and strengths for a collection of N vortices and then make repeated use of either Eq. (11) or ((22) to compute the net velocity at the center of each due to all others in the system. The positions are then adjusted by integrating Eq. (2) over a small time interval Δt. The procedure is then repeated over and over again until the desired final time is reached. While this can work in principle, experience has shown that two vortices may move into close proximity of one another, in which case extremely large velocities may result. Large velocities require small time steps for accurate integration and will also often result in unwanted orbiting of two or more vortices on very small length scales. The origin of these problems lies in the singular nature of the idealized point vortex approximation, which predicts unbounded velocities as the center is approached. Of course infinite velocities are not possible in reality as viscous friction will always ensure that the velocity field is smooth at some scale. Modification of the vortex velocity field in order to remove the non-physical singularity is discussed in the following section.

3 Smooth vortex approximations

While the inviscid approximation allows the velocity and vorticity to become infinite at the vortex center, viscosity in the real world will prevent this from happening. Thus for both physical and practical reasons, it is necessary to modify the formulation in order to remove the singularity. While there are many different ways to approach such a modification, perhaps the simplest is to replace the inviscid vorticity delta function with a non-singular approximation to it. If $d(r)$ is such an approximation, then we take

$$\omega(r) = \Gamma d(r). \tag{23}$$

In making this approximation, it is advantageous to preserve a few key integral properties of the inviscid approximation. Since the inviscid approximation amounts to a delta function vorticity distribution, we have the following relation for its moments [1]

$$\int_0^\infty r^{2n} d(r) 2\pi r dr = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases} \tag{24}$$

While it will be impossible to achieve zero moments for all $n > 0$, it may be possible to at least make the first few vanish.

Our approximation in Eq. (23) can be substituted into the definition of the vorticity (Eq. (12)) in order to yield the corresponding azimuthal velocity distribution:

$$\omega(r) = \frac{1}{r}\frac{d}{dr}(ru_\theta) = \Gamma d(r) \tag{25}$$

$$u_\theta(r) = \frac{\Gamma}{r}\int_0^r \xi d(\xi) d\xi \tag{26}$$

Now we can consider different choices for the function $d(r)$ and determine the corresponding velocity distributions. A few examples are

the top-hat:

$$d(r) = \begin{cases} \frac{1}{\pi\Delta^2} & r \le \Delta \\ 0 & r > \Delta \end{cases} \tag{27}$$

$$u_\theta(r) = \begin{cases} \frac{\Gamma}{2\pi r}\left(\frac{r}{\Delta}\right)^2 & r \le \Delta \\ \frac{\Gamma}{2\pi r} & r > \Delta \end{cases} \tag{28}$$

the cone:

$$d(r) = \begin{cases} \frac{3}{\pi a^2}\left(1 - \frac{r}{a}\right) & r \le a \\ 0 & r > a \end{cases} \tag{29}$$

$$u_\theta(r) = \begin{cases} \frac{\Gamma}{2\pi r}\left[3\left(\frac{r}{a}\right)^2 - 2\left(\frac{r}{a}\right)^3\right] & r \le a \\ \frac{\Gamma}{2\pi r} & r > a \end{cases} \qquad a = \sqrt{\frac{5}{3}}\Delta \tag{30}$$

polynomial:

$$d(r) = \begin{cases} \frac{2}{\pi b^2}\left[1 - 2\left(\frac{r}{b}\right)^2 + \left(\frac{r}{b}\right)^4\right] & r \le b \\ 0 & r > b \end{cases} \tag{31}$$

$$u_\theta(r) = \begin{cases} \frac{\Gamma}{2\pi r}\left[3\left(\frac{r}{b}\right)^2 - 3\left(\frac{r}{b}\right)^4 + \left(\frac{r}{b}\right)^6\right] & r \le b \\ \frac{\Gamma}{2\pi r} & r > b \end{cases} \qquad b = \sqrt{2}\Delta \tag{32}$$

[1] More precisely, we should weight the integral with the factors $x^i y^j$ with $m = i + j$ and also integrate over θ from 0 to 2π. If we then use the fact that $x = r\cos\theta$, $y = r\sin\theta$, we see that the the integrals for odd m vanish due to symmetry in the $\cos^m\theta$ and $\cos^m\theta$ terms. The end result is that we only require the even moments to vanish.

the Gaussian

$$d(r) = \frac{2}{\pi\Delta^2} \exp\left[-2\left(\frac{r}{\Delta}\right)^2\right] \tag{33}$$

$$u_\theta(r) = \frac{\Gamma}{2\pi r}\left\{1 - \exp\left[-2\left(\frac{r}{\Delta}\right)^2\right]\right\} \tag{34}$$

the 4$^{\text{th}}$ order Gaussian

$$d(r) = \frac{1}{2\pi\sigma^2}\left\{4\exp\left[-\left(\frac{r}{\sigma}\right)^2\right] - \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right]\right\} \tag{35}$$

$$u_\theta(r) = \frac{\Gamma}{2\pi r}\left\{1 - 2\exp\left[-\left(\frac{r}{\sigma}\right)^2\right] + \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right]\right\} \qquad \sigma \simeq \Delta \tag{36}$$

With the exception of the 4$^{\text{th}}$ order Gaussian, all of these functions are constructed so that they have identical second moments, and so that the parameter Δ is an estimate for the width of the distribution. Since the 4$^{\text{th}}$ order Gaussian has zero second moment, we must use some other measure of its width. While several possibilities exist, here we simply note that by taking $\sigma = \Delta$ the resulting velocity profile is in reasonable agreement with the other profiles.

It is of interest to determine the effective delta function approximation associated with the Krasny scheme. When written in Cartesian coordinates, Krasny simply added the parameter ϵ^2 to the denominator in the expressions for the vortex velocity, i.e.

$$u \simeq -\frac{\Gamma y}{2\pi(x^2+y^2+\epsilon^2)} \qquad v \simeq \frac{\Gamma x}{2\pi(x^2+y^2+\epsilon^2)} \tag{37}$$

When written in polar coordinates, these approximations are equivalent to

$$u_\theta \simeq \frac{\Gamma}{2\pi r}\left[\frac{1}{1+\left(\frac{\epsilon}{r}\right)^2}\right] \tag{38}$$

If this expression is substituted into Eq. (26) and the derivative taken we find

$$d(r) = \frac{1}{\pi\epsilon^2}\left[1+\left(\frac{r}{\epsilon}\right)^2\right]^{-2} \qquad \text{Krasny} \tag{39}$$

This expression decays like r^{-4} for large r. While this would seem sufficiently fast, the second moment diverges since the integral is weighted by $r^2 2\pi r dr \sim r^3 dr$, leaving the integrand with a net $1/r$ for large r. While this may not be a large concern in practice, it prevents assigning a distribution width based on the second moment. The first moment does converge, and if this is equated to that of the Gaussian, then one obtains the relation $\epsilon = \Delta/\sqrt{2\pi} \simeq 0.4\Delta$. Visually, the velocity profile looks more consistent with the others if $\epsilon \simeq 0.5\Delta$. The velocity profiles associated with the various approximations to the delta function are plotted below

Up to this point, our expressions for the smoothed vortex velocity distributions are valid for non-periodic boundary conditions. In order to determine periodic boundary condition counterparts, a procedure similar to that used in section 2 must be followed. Although it would be ideal to use the conformal mapping technique, the fact that the vorticity is now distributed over a finite region in space precludes writing solutions that satisfy Laplace's equation. We could ignore this detail and proceed nonetheless in order to achieve an approximate result, but perhaps this is not the best approach. The more accurate approach would be to return to the idea of an infinite sum over the periodic image vortices as in Eq. (17). Prior to doing this, it is advantageous to write each of the various vortex velocity distributions in the unified form

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - f(r)\right] \tag{40}$$

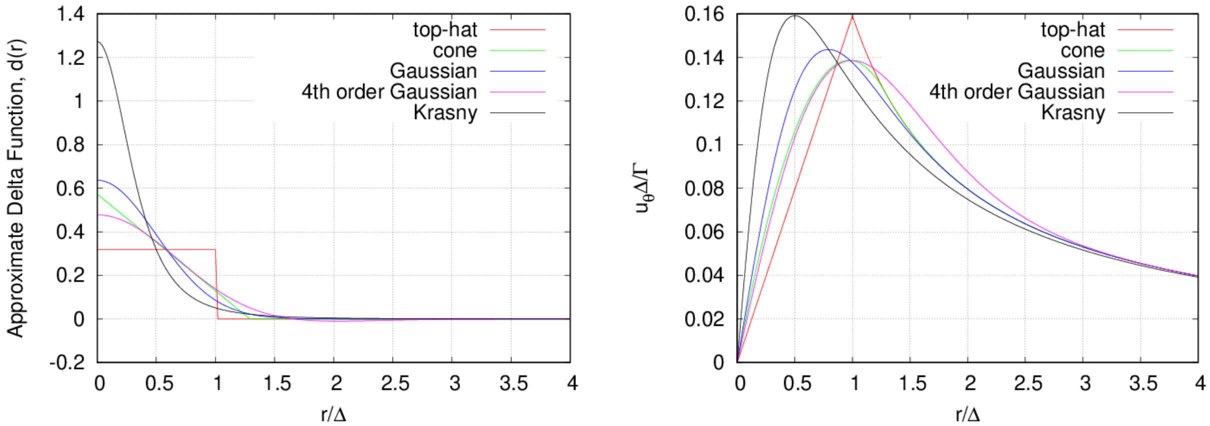

Figure 1: Approximate delta functions and resulting vortex velocity profiles. The 4th order Gaussian result is plotted for $\sigma = \Delta$ and the Krasny result is plotted for $\epsilon = 0.5\Delta$.

with

$$
\begin{aligned}
f(r) &= \begin{cases} 1 - \left(\frac{r}{\Delta}\right)^2 & r \le \Delta \\ 0 & r > \Delta \end{cases} && \text{Top hat} \\
f(r) &= \begin{cases} 1 - 3\left(\frac{r}{a}\right)^2 + 2\left(\frac{r}{a}\right)^3 & r \le a \\ 0 & r > a \end{cases} && \text{Cone} \\
f(r) &= \begin{cases} 1 - 3\left(\frac{r}{b}\right)^2 + 3\left(\frac{r}{b}\right)^4 - \left(\frac{r}{b}\right)^6 & r \le b \\ 0 & r > b \end{cases} && \text{Polynomial} \\
f(r) &= \exp\left[-2\left(\frac{r}{\Delta}\right)^2\right] && \text{Gaussian} \\
f(r) &= 2\exp\left[-\left(\frac{r}{\sigma}\right)^2\right] - \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right] && \text{4th order Gaussian} \\
f(r) &= \left[1 + \left(\frac{r}{\epsilon}\right)^2\right]^{-1} && \text{Krasny}
\end{aligned}
\tag{41}
$$

These functions are plotted in Figure 2.
If Eq (40) is substituted into Eq. (17) it is possible to write

$$
W_j(z) = u_j + iv_j = -\frac{i\Gamma_j}{2\pi}\left\{\sum_{n=-\infty}^{\infty}\frac{1}{z-(z_j+nx_L)} - \sum_{n=-\infty}^{\infty}\frac{f(|z-(z_j+nx_L|)}{z-(z_j+nx_L)}\right\} \tag{42}
$$

Note that, even though the smoothed vortex solutions do not satisfy Laplace's equation, it is still possible to use complex notation for the velocity components. The first sum above is equivalent to the cot term as in Eq. (18) whereas the second sum is unlikely to equate to a simple analytic function. Not to worry, however, since Figure 2 shows that all of the functions $f(r)$ decay to zero rapidly as r becomes large. While rapid decay is not as pronounced for the Krasny function, this is not a concern since we have an ad hoc closed form solution for this case where ϵ^2 is simply added to the denominator in Eq. (22). For all other cases we can simply truncate the second sum above at a few term, say $n \le m$. Thus, making use of Eq. (22) for the first sum and truncating the second at

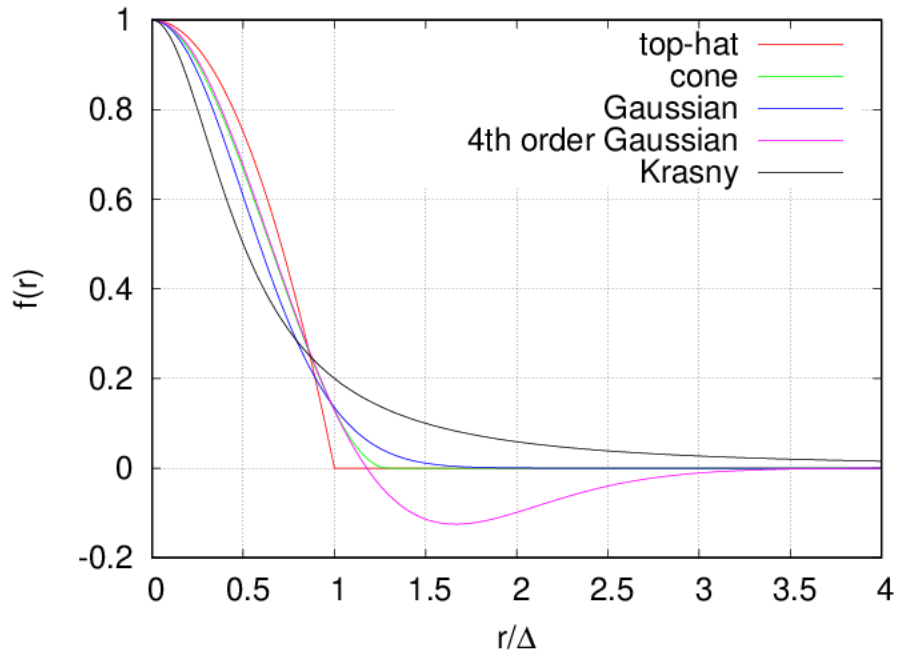

Figure 2: Damping functions associated with the various approximations to the delta function. The 4th order Gaussian result is plotted for $\sigma = \Delta$ and the Krasny result is plotted for $\epsilon = 0.5\Delta$.

$\pm m$ allows us to write

$$(u_j, v_j) = \frac{\Gamma_j}{2x_L}\left(\frac{-\sinh\left[\frac{2\pi(y-y_j)}{x_L}\right], \quad \sin\left[\frac{2\pi(x-x_j)}{x_L}\right]}{\cosh\left[\frac{2\pi(y-y_j)}{x_L}\right] - \cos\left[\frac{2\pi(x-x_j)}{x_L}\right]}\right) - \frac{\Gamma_j}{2\pi}\sum_{n=-m}^{m}\frac{-(y-y_j), \quad [x-(x_j+nx_L)]}{[x-(x_j+nx_L)]^2+(y-y_j)^2} f\left(\sqrt{[x-(x_j+nx_L)]^2+(y-y_j)^2}\right) \quad (43)$$

Now it only remains to determine the sum truncation index m. First note from Figure 2 that $f(r)$ is essentially zero for $r > b\Delta$, where b is a order 1 number that depends somewhat on the approximation. For example $b = 1$, $\sqrt{(5/3)}$, ~ 1.75, ~ 3.25 for the Top hat, Cone, Gaussian, and 4th order Gaussian approximations respectively. Now if we call the argument of the damping function f above $r_j(n)$, we see that it is possible to truncate the sum if

$$r_j(n) = \sqrt{[x-(x_j+nx_L)]^2+(y-y_j)^2} \geq b\Delta, \quad (44)$$

which can be rearranged to give

$$n \leq \text{int}\left\{\left(\frac{x-x_j}{x_L}\right) \pm \sqrt{\left(\frac{b\Delta}{x_L}\right)^2 - \left(\frac{y-y_j}{x_L}\right)^2}\right\} \qquad \text{for} \qquad (y-y_j) < b\Delta \quad (45)$$

where int() is the integer truncation. At this point it is useful to consider a few limiting cases. First note that Eq. (44) is satisfied for all n if $(y-y_j) > b\Delta$, and hence the restriction on Eq. (45). Since Eq. (44) is satisfied independent of n in this case, no terms are required in the sum. If $(y-y_j) < b\Delta$ then Eq. (45) can be used to determine the lower and upper limits on the sum. In this case it is useful to note that the term $(x-x_j)/x_L$ must be in the range $[0:1]$. In addition, the smoothing length scale will generally be a small fraction of the periodicity length and, consequently, the term

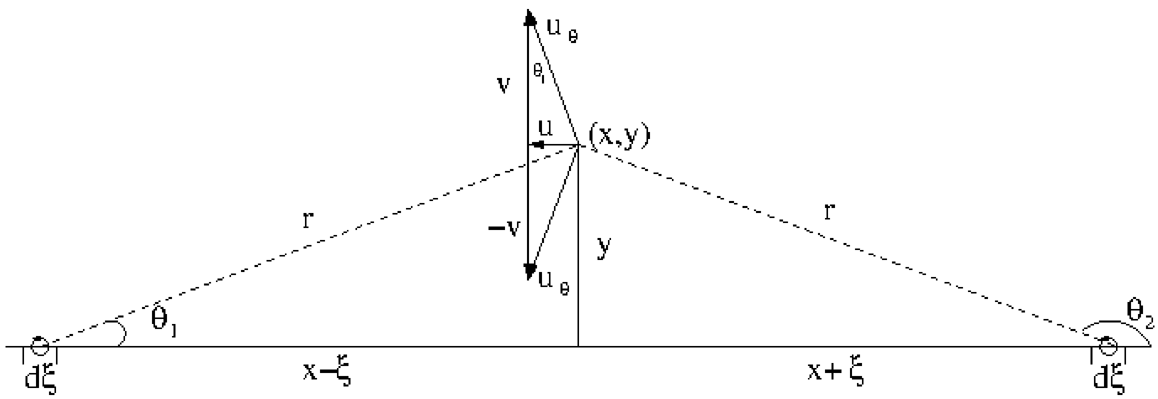

Figure 3: Contributions to the u velocity component by vortex sheet segments of length $d\xi$ at positions $x \pm \xi$.

$b\Delta/x_L$ should rarely exceed 1. The most restrictive case is for $(y - y_j)/x_L = 0$, in which case Eq. (45) reduces to

$$n \leq \mathrm{int}\left[\left(\frac{x - x_j}{x_L}\right) \pm \frac{b\Delta}{x_L}\right] \tag{46}$$

If we now take the pessimistic estimate $1 \leq b\Delta/x_L < 2$, then when $(x - x_j)/x_L = 0$ we achieve the lower bound $n = -1$ and when $(x - x_j)/x_L = 1$ the upper bound $n = 2$. While these limits could be implemented in Eq. (43), a slight modification can be used to reduce the number of terms in the sum. To accomplish this note that the first term in Eq. (43) is invariant under a shift by the periodicity length in the x-direction, i.e. $\tilde{x}_j = x_j + x_L$. If we undertake this transformation when $(x-x_j)/x_L > 1/2$ then we are simply moving our reference point to the closest vortex in the periodic array. If we do this, then the quantity $(x - x_j)/x_L$ will be contained in the range $[-0.5 : 0.5]$ and the limits on n will fall in the symmetric bound $[-m : m]$ with

$$m = \mathrm{int}\left[\frac{1}{2} + \frac{b\Delta}{x_L}\right] \qquad \text{for } x_j \text{ symmetrized} \tag{47}$$

If this approach is adopted then we only require a single term at $n = 0$ if $b\Delta/x_L < 1/2$, and three terms at $n = -1,\ 0,\ 1$ if $1/2 < b\Delta/x_L < 3/2$. Additional terms should never really be needed since the smoothing distance either exceeds x_L or at least becomes a large fraction of it when $b\Delta/x_L > 3/2$.

4 Velocity profile due to a flat vortex sheet

It it of interest to compute the velocity profile taken on a line passing normal to a vortex sheet. Although vortex methods will always use a finite number of vortices, it is easier to perform the analysis for a continuous sheet composed of an infinite number of vortices. Within this framework we characterize the vortex strength by its circulation per unit length γ, which we will assume is a constant in the analysis below. To begin, imagine that the entire x-axis is covered with a vortex sheet of strength γ. We then strive to determine an expression for the resulting horizontal velocity component as a function of the distance normal to the sheet, i.e. $u(y)$ (there can be no explicit x dependence since the sheet is infinite in that direction). The situation is shown in Figure 3, where the contributions to $u(y)$ due to two vortex sheet segments, both of length $d\xi$ and located at $x \pm \xi$, $y = 0$ is diagramed. Each vortex patch induces a velocity du_θ normal to vector between the patch and the point (x, y). If these two du_θ components are decomposed into u and v components, we see that the v components cancel, whereas the horizontal components superimpose. Thus the velocity due to the sheet is strictly in the horizontal direction and it is made up of equal contributions due to the two halves of the sheets lying to either side of the point x. With the contribution to the horizontal component being $du = -(du_\theta/d\xi)\sin\theta d\xi = -\tilde{u}_\theta \sin\theta d\xi$, the net velocity do to the entire sheet is

$$u = -\int_{-\infty}^{\infty} \tilde{u}_\theta \sin\theta d\xi \tag{48}$$

where $\tilde{u}_\theta = du_\theta/d\xi$ is the induced velocity per unit length of the sheet. Since $\tilde{u}_\theta = \tilde{u}_\theta(r)$ it is useful to switch the variable of integration from ξ to r. This can be accomplished by noting that $x - \xi = \sqrt{(r^2 - y^2)}$ and consequently $d\xi = -r/\sqrt{(r^2 - y^2)}dr$. If this result is substituted into the integral above and the fact that $y = r\sin\theta$ is used, we get

$$u(y) = -2\int_{y}^{\infty} \tilde{u}_\theta \left[\frac{y}{\sqrt{(r^2 - y^2)}}\right] dr \tag{49}$$

where the factor of 2 accounts for the contributions of both halves of the sheet as r varies between 0 and ∞. Also the limits of integration have been swapped and the sign changed due to the fact that r varies between ∞ and y as ξ varies from $-\infty$ to 0.

We can produce yet another form with $d\theta$ as the integration variable by noting

$$x - \xi = r\cos\theta \qquad - d\xi = \cos\theta dr - r\sin\theta d\theta \tag{50}$$

$$y = r\sin\theta \qquad dy = \sin\theta dr - r\cos\theta d\theta = 0 \tag{51}$$

where it has been noted that $dy = 0$ (since y is constant as the integral is taken). If the second equation is solved for dr and the result substituted back into the first, we find $\sin\theta d\xi = r d\theta$, and thus the integral for $u(y)$ can also be written as

$$u(y) = -\int_{\theta_a}^{\theta_b} \tilde{u}_\theta \, r \, d\theta \tag{52}$$

This latter form is the most convenient for computing the velocity profile due to an inviscid vortex sheet where $\tilde{u}_\theta = \gamma/(2\pi r)$:

$$u(y) = -\int_{\theta_a}^{\theta_b} \left(\frac{\gamma}{2\pi r}\right) r \, d\theta \tag{53}$$

$$= -\frac{\gamma}{2\pi}\int_{\theta_a}^{\theta_b} d\theta \tag{54}$$

$$= -\frac{\gamma}{2\pi}(\theta_b - \theta_a) \tag{55}$$

If $y > 0$, then θ varies between 0 and π, and we have $u(y) = -\gamma/2$. Conversely if $y < 0$, then the angle varies between 0 and $-\pi$ and we have $u(y) = \gamma/2$. Thus the velocity profile is a step function with jump γ when crossing the sheet.

Equation (52) is also the most convient form for vortex velocity profiles involving powers of r. If $\tilde{u}_\theta \sim r^n$, then $\tilde{u}_\theta r \sim y^{n+1}/\sin^{n+1}\theta$. To illustrate this, consider the top-hat approximation, which results in the velocity profile of Eq. (28). Assuming $0 < y < \Delta$, making use of Eq. (52), and symmetry between the left and right halves of the sheet, we have

$$u(y) = -2\left[\frac{\gamma}{2\pi}\int_0^{\theta_\Delta} d\theta + \frac{\gamma y^2}{2\pi\Delta^2}\int_{\theta_\Delta}^{\pi/2} \frac{1}{\sin^2\theta} d\theta\right] \qquad \text{for } 0 \le y \le \Delta \tag{56}$$

where $\theta_\Delta = \tan^{-1}(y/\sqrt{\Delta^2 - y^2})$ is the angle when $r = \Delta$. The second integral evaluates to $-\cot\theta$ and thus

$$\text{Top}-\text{hat}: \qquad u(y) = \begin{cases} -\frac{\gamma}{\pi}\left[\tan^{-1}\left(\frac{y}{\sqrt{\Delta^2 - y^2}}\right) + \left(\frac{y}{\Delta}\right)\sqrt{1 - (y/\Delta)^2}\right] & \text{for } 0 \le y \le \Delta \\ -\frac{\gamma}{2} & \text{for } y > \Delta \end{cases} \tag{57}$$

Using a similar approach, the resulting velocity profile for the cone approximation is

$$\text{Cone}: \quad u(y) = \begin{cases} -\frac{\gamma}{\pi}\left[\tan^{-1}\left(\frac{y}{\sqrt{a^2-y^2}}\right) + \frac{1}{2}\left(\frac{y}{a}\right)^3 \ln\left(\frac{a-\sqrt{a^2-y^2}}{a+\sqrt{a^2-y^2}}\right) + 2\left(\frac{y}{a}\right)\sqrt{1-(y/a)^2}\right] & \text{for } 0 \le y \le a \\ -\frac{\gamma}{2} & \text{for } y > a \end{cases} \tag{58}$$

and for the polynomial approximation

$$\text{Polynomial}: \quad u(y) = \begin{cases} -\frac{\gamma}{\pi}\left\{\tan^{-1}\left(\frac{y}{\sqrt{b^2-y^2}}\right) + \left(\frac{y}{b}\right)\sqrt{1-\left(\frac{y}{b}\right)^2}\left[\frac{1}{5}\left(1-\left(\frac{y}{b}\right)^2\right)^2 - \left(1-\frac{2}{3}\left(\frac{y}{b}\right)^2\right)\left(1-\left(\frac{y}{b}\right)^2\right) + \left(3-3\left(\frac{y}{b}\right)^2+\left(\frac{y}{b}\right)^4\right)\right]\right\} & \text{for } 0 \le y \le b \\ -\frac{\gamma}{2} & \text{for } y > b \end{cases} \tag{59}$$